Local Fields

Dr. T. Dokchitser

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LaTeXed by Sebastian Pancratz

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These notes have not been checked by Dr. T. Dokchitser and should not be regarded as official notes for the course. In particular, the responsibility for any errors is mine — please email Sebastian Pancratz (`sfp25`) with any comments or corrections.

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Chapter 1

Valued Fields

An *absolute value* on a field K is a function $|\cdot|\colon K \to \mathbb{R}_{\geq 0}$ such that

(i) $|x| = 0$ if and only if $x = 0$;
(ii) $|xy| = |x||y|$;
(iii) $|x + y| \leq |x| + |y|$.

Example. Take $K = \mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$ and let $|\cdot|$ be the usual absolute value, which we will denote $|\cdot|_\infty$.

Example. For any field K, let

$$|x| = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases}$$

called the *trivial absolute value* on K.

From (i) and (ii) it follows that $|1| = |-1| = 1$. Generally, if $x \in K$ such that $x^n = 1$ then $|x| = 1$. In particular, if $K = \mathbb{F}_{p^n}$ is a finite field, or $K = \bar{\mathbb{F}}_p$, then K has only trivial absolute value. Also note that $|x/y| = |x|/|y|$ for $y \neq 0$ and $|x^n| = |x|^n$.

Example (p-adic Absolute Value on $\mathbb{Q}$)**.** Fix a prime p and $0 < \alpha < 1$. Write $x \in \mathbb{Q}^*$ as

$$x = p^n \frac{a}{b}$$

with $n \in \mathbb{Z}$ and a, b coprime to p. Then define $|x| = |p^n a/b| = \alpha^n$. This is called the *$p$-adic absolute value* on $\mathbb{Q}$. It is indeed an absolute value, for if $x = p^n a/b$, $y = p^m c/d$ then

$$\begin{aligned} |xy| &= |p^{n+m}\frac{ac}{bd}| = \alpha^{m+n} = |x||y| \\ |x + y| &= |p^{\min\{n,m\}}\frac{\star}{bd}| = |p^{\min\{n,m\}}||\frac{\star}{bd}| \\ &\leq \alpha^{\min\{n,m\}} = \max\{|x|, |y|\} \leq |x| + |y| \end{aligned}$$

So a rational number is small with respect to $|\cdot|$ if and only if it is divisible by a large power of p. To remove ambiguity in the choice of α, we make the following definition.

Definition. Two absolute values $|\cdot|$, $\|\cdot\|$ on K are *equivalent* if there exists $c > 0$ such that

$$|x| = \|x\|^c$$

for all $x \in K$. The *normalised p-adic absolute value* is the one with $\alpha = 1/p$ and it is denoted $|\cdot|_p$.

Example. Let $p = 5$. Then

$$|5^n|_5 = 5^{-n} \qquad |10|_5 = \frac{1}{5}$$

$$|\frac{1}{10}|_5 = 5 \qquad |\frac{2}{3}|_5 = 1$$

If an absolute value on K satisfies the following stronger condition (iii')

$$|x + y| \leq \max\{|x|, |y|\}$$

we call it *ultrametric* or *non-Archimedean*. Otherwise, we say it is *Archimedean*.

Theorem 1.1 (Ostrowski)**.** Any non-trivial absolute value $|\cdot|$ on $\mathbb{Q}$ is equivalent to either $|\cdot|_\infty$ or $|\cdot|_p$ for some p.

Proof. Let $a, b > 1$ be integers and write b^n in base a,

$$b^n = c_m a^m + c_{m-1} a^{m-1} + \cdots + c_0$$

with $c_i \in [0, a-1]$. Let $M = \max\{|1|, \ldots, |a-1|\}$. Then

$$\begin{aligned} |b^n| &\leq |c_m||a|^m + \cdots + |c_0| \\ &\leq (m+1)M \max\{|a|^m, \ldots, |1|\} \\ &\leq (n \log_a b + 1) M \max\{1, |a|^m\} \end{aligned}$$

Taking nth roots and letting $n \to \infty$,

$$|b| \leq \max\{1, |a|^{\log_a b}\} \tag{$*$}$$

Case 1. Assume $|b| > 1$ for some integer $b > 1$. By $(*)$,

$$|b| \leq \max\{1, |a|^{\log_a b}\} = |a|^{\log_a b}$$

so $|a| > 1$ for all $a > 1$. Interchanging a and b in $(*)$,

$$|a| \leq |b|^{\log_b a}$$

so

$$|b|^{\frac{1}{\log b}} = |a|^{\frac{1}{\log a}}$$

Equivalently, $|a| = a^\lambda$ for all $a \geq 1$ and some λ independent of a, so $|\cdot| \sim |\cdot|_\infty$.

Case 2. Suppose $|b| \leq 1$ for all integers $b \geq 1$. Then there is a $b > 1$ such that $|b| < 1$, otherwise $|\cdot|$ is trivial. Take such a b and write $b = p_1^{n_1} \cdots p_k^{n_k}$. Then

$$1 > |b| = |p_1|^{n_1} \cdots |p_k|^{n_k}$$

so there exists p such that $|p| < 1$. It suffices to show that $|q| = 1$ for all primes $q \neq p$, and it then follows that $|\cdot| \sim |\cdot|_p$.

Suppose $|p| < 1$ and $|q| < 1$ for some $p \neq q$. Take $n, m \geq 1$ such that $|p^n| < 1/2$, $|q^m| < 1/2$. As p^n, q^m are coprime, $1 = xp^n + yq^m$ for some $x, y \in \mathbb{Z}$, so

$$1 \leq |x||p^n| + |y||q^m| < \frac{1}{2} + \frac{1}{2} = 1$$

contradiction. $\square$

We do not need the following result, but there is a complete classification of Archimedean absolute values.

Theorem 1.2. If $|\cdot|$ is an Archimedean absolute value on a field K then there exists an injection $K \hookrightarrow \mathbb{C}$ such that $|\cdot| \sim |\cdot|_\infty$ on $\mathbb{C}$.

Chapter 2

Non-Archimedean Absolute Values and Valuations

From now on, $|\cdot| \colon K \to \mathbb{R}_{\geq 0}$ is non-Archimedean and non-trivial. We sometimes say K is non-Archimedean with a fixed $|\cdot|$ in mind.

Pick $0 < \alpha < 1$ and write $|x| = \alpha^{v(x)}$, i.e., let $v(x) = \log_\alpha |x|$.

$$K^* \to \mathbb{R}_{>0} \xrightarrow{\log_\alpha} (\mathbb{R}, +)$$

Then $v(x)$ is a *valuation*, an additive version of $|\cdot|$.

Definition. The map $v \colon K^* \to \mathbb{R}$ is a *valuation* if

(i) $v(K^*) \neq \{0\}$;
(ii) $v(xy) = v(x) + v(y)$;
(iii) $v(x+y) \geq \min\{v(x), v(y)\}$.

Valuations v and cv, for $c > 0$ a real constant, are called *equivalent*. A valuation determines a non-trivial non-Archimedean absolute value and vice versa.

We extend v to K formally by letting $v(0) = \infty$. The image $v(K^*)$ is an additive subgroup of $\mathbb{R}$, the *value group* of v. If it is discrete, i.e., isomorphic to $\mathbb{Z}$, we say v is a *discrete valuation*. If $v(K^*) = \mathbb{Z}$, we call v *normalised discrete valuation*.

We will only study discrete valuations.

Example. Let $K = \mathbb{Q}$ and p a prime. Then

$$v_p = \mathrm{ord}_p \colon \mathbb{Q}^* \ni p^n \frac{a}{b} \mapsto n \in \mathbb{Z}$$

is the p-adic valuation,

$$|x|_p = \left(\frac{1}{p}\right)^{v_p(x)}$$

Alternatively,

$$v_p(x) = \max_{r \in \mathbb{Z}} \{r : x \in p^r \mathbb{Z}\}$$

Generally, if K is a number field with $[K : \mathbb{Q}] < \infty$, let $\{0\} \leq P \subset \mathcal{O}_K$ be a prime ideal. Then define

$$v_P \colon K \to \mathbb{Z}, x \mapsto \max_{r \in \mathbb{Z}} \{r : x \in P^r \mathcal{O}_K\}$$

This is a normalised discrete valuation and every valuation on K is of this form, i.e., there is an analogue of Ostrowski's theorem for number fields.

Example. Let $K = k(t)$. Define

$$v_0\left(t^n \frac{p(t)}{q(t)}\right) = n$$

where $p(0), q(0) \neq 0$. This is a normalised discrete valuation, the order of zeros or poles at $t = 0$. In fact, for any $a \in K$ we may define

$$v_a\left((t-a)^n \frac{p(t)}{q(t)}\right) = n$$

where $p(a), q(a) \neq 0$. For instance, let $f(t) = t^2(t-1)/(t-2)^5$. Then

$$v_0(f) = 2 \qquad v_1(f) = 1 \qquad v_2(f) = -5 \qquad v_a(f) = 0$$

for all other $a \in k$. There is also

$$v_\infty\left(\frac{p(t)}{q(t)}\right) = \deg q(t) - \deg p(t) \in \mathbb{Z}$$

which again is a valuation, called the order at ∞.

If $X = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere let $K = \mathbb{C}(z)$ be the field of meromorphic functions on X. The above valuation is

$$v_a(f) = \operatorname{ord}_{z=a} f(z)$$

for every $a \in X$, including ∞.

If $k = \mathbb{C}$, or in general, an algebraically closed field, then these are the only valuations on $K = k(t)$ with $v(k^*) = \{0\}$.

2.1 Aside on Algebraically Closed Fields

Definition. A field K is algebraically closed if the following equivalent conditions are satisfied.

(i) Every polynomial of degree n over K has precisely n roots, counted with multiplicity.
(ii) Every non-constant polynomial is a product of linear factors.
(iii) If $f \in K[X]$ is non-constant and irreducible then f is linear.
(iv) K has no non-trivial finite extensions.

Example. $\mathbb{C}$ is algebraically closed. $\mathbb{Q}, \mathbb{R}, \mathbb{F}_{p^n}$, $k(t)$ are not algebraically closed.

Theorem 2.1. Let K be any field. Then there exists an algebraically closed field $\bar{K}$ unique up to isomorphism such that $K \subset \bar{K}$ and every element of $\bar{K}$ is algebraic over K. $\bar{K}$ is called an *algebraic closure*.

Example. $\mathbb{C}$ is algebraically closed so $\bar{\mathbb{C}} = \mathbb{C}$. $\bar{\mathbb{R}} = \mathbb{C}$. $\bar{\mathbb{Q}}$ is the set of $\alpha \in \mathbb{C}$ satisfying polynomials with rational coefficients.

Exercise 1. Show that $\bar{\mathbb{Q}}$ as defined above is an algebraically closed field. Further show that there exists a sequence $(K_n)_{n\geq 1}$ of finite Galois extensions $Q \subset K_1 \subset K_2 \subset \cdots$ such that $\bar{\mathbb{Q}} = \bigcup_{n\geq 1} K_n$.

2.2 Algebraic Properties of Valuations

Let $v\colon K^* \to \mathbb{R}$ be a valuation corresponding to the absolute value $|\cdot|\colon K \to \mathbb{R}_{\geq 0}$. Then

$$\mathcal{O} = \mathcal{O}_v = \mathcal{O}_K = \{x \in K : v(x) \geq 0\} = \{x \in K : |x| \leq 1\}$$

is a ring, called the *valuation ring* of v. K is its field of fractions, and

$$x \in K \setminus \mathcal{O} \implies \frac{1}{x} \in \mathcal{O}$$

The set of units in $\mathcal{O}$ is

$$\mathcal{O}^\times = \{x \in K : v(x) = 0\} = \{x \in K : |x| = 1\}$$

and

$$M = \{x \in K : v(x) > 0\} = \{x \in K : |x| < 1\}$$

is an ideal in $\mathcal{O}$. Because $\mathcal{O} = \mathcal{O}^\times \cup M$, M is a unique maximal ideal, so $\mathcal{O}$ is *local*. $k = \mathcal{O}/M$ is a field, called the *residue field* of v or of K.

Suppose $v\colon K^* \twoheadrightarrow \mathbb{Z}$ is normalised discrete. Take $\pi \in M$ with $v(\pi) = 1$, called a *uniformiser*. Then every $x \in K^*$ can be written uniquely as

$$x = u\pi^n$$

for a unit $u \in \mathcal{O}^\times$ and $n \in \mathbb{Z}$. Every $x \in \mathcal{O}$ can be written uniquely as

$$x = u\pi^n$$

for a unit $u \in \mathcal{O}^\times$ and $n \in \mathbb{Z}_{\geq 0}$. Every $x \in M$ can be written uniquely as

$$x = u\pi^n$$

for a unit $u \in \mathcal{O}^\times$ and $n \geq 1$. In particular, $M = (\pi)$ is principal. Moreover, every ideal $I \subset \mathcal{O}$ is principal,

$$\mathcal{O} \supset I \neq (0) \implies I = (\pi^n)$$

where $n = \min\{v(x) : x \in I\}$, so $\mathcal{O}$ is a principal ideal domain (PID).

Example. Let $K = \mathbb{Q}$, $v = v_p$. Then

$$\mathcal{O} = \left\{\frac{x}{y} : (y, p) = 1\right\}$$
$$M = \left\{\frac{x}{y} : (y, p) = 1, p \mid x\right\} = (p)\mathcal{O}$$
$$\mathcal{O}/M = k \cong \mathbb{F}_p, \frac{x}{y} \mapsto \frac{x \mod p}{y \mod p}$$

Example. Let $K = k(T)$ and consider v_a for some $a \in k$. Then

$$\mathcal{O} = \left\{\frac{f}{g} : g(a) \neq 0\right\}$$
$$M = \left\{\frac{f}{g} : g(a) \neq 0, f(a) = 0\right\}$$
$$\mathcal{O}/M \xrightarrow{\sim} k, f \mapsto f(a)$$

the evaluation map.

Definition. A *discrete valuation ring* (DVR) is a local integral PID, which is not a field.

Theorem 2.2. (i) Suppose $v\colon K^* \twoheadrightarrow \mathbb{Z}$ is a valuation. Then $\mathcal{O}_v$ is a DVR.

(ii) If R is a DVR then there exists a unique valuation v on its field of fractions K such that $R = \mathcal{O}_v$.

Proof. (i) $\mathcal{O}_v \subset K$ is a subring, hence is an integral domain; we have already shown it is local and a PID. $\pi^{-1} \in K \setminus \mathcal{O}$, so $\mathcal{O}$ is not a field.

(ii) Let R be a DVR. R is local so has a unique maximal ideal M, and R is a PID so $M = (\pi)$ for some $\pi \in R$. (Recall that if R is a PID then R is also a UFD.) If π' is another irreducible element then (π') is maximal, hence $(\pi) = (\pi')$ so π' is associate to π. As R is a UFD, every element is uniquely of the form $u\pi^n$, $u \in R$ a unit.

Now define v on R by letting $v(u\pi^n) = n$, and extend to K by $v(x/y) = v(x) - v(y)$. Check it is a valuation.

□

Definition. Let $R \subset S$ be rings. Then $x \in S$ is *integral* over R if

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

for some $a_i \in R$. The *integral closure* of R in S is

$$\{x \in S : x \text{ integral over } R\}$$

This is a ring, contained in S and containing R.

Example. Let $R = \mathbb{Z}$, $S = \mathbb{C}$. Then the integral closure is the ring of algebraic integers.

A domain R is *integrally closed* if R is its integral closure in its field of fractions. Equivalently, for all $y \in \mathrm{Frac}(R)$, y is integral over R if and only if $y \in R$.

Theorem 2.3. Let R be a domain. Then R is a DVR if and only if R is Noetherian, integrally closed, and has a unique non-zero prime ideal.

Proof. Suppose R is a DVR. We know every ideal is principal and hence finitely generated, so R is Noetherian.

Now take $x \in \mathrm{Frac}(R) = K$, $x \notin R$, i.e., $v(x) = m < 0$, but satisfying

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

Then

$$mn = v(x^n) = v(-x^n) = v(a_{n-1}x^{n-1} + \cdots + a_1 x + a_0) \geq m(n-1)$$

a contradiction.

Every ideal $(0) \neq I \subset R$ is of the form $I = (\pi^k)$, and (π^m) is prime if and only if $m = 1$. Thus (π) is the unique non-zero prime ideal. This completes one direction of the proof.

Lemma 2.4 (Poor Man's Factorisation)**.** Let R be Noetherian and $I \subset R$ an ideal. Then there exists prime ideals $P_1, \ldots, P_n$ such that

$$I \subset P_i, \qquad \prod_{i=1}^{n} P_i \subset I$$

Proof. Let $\mathcal{S}$ be the set of ideals I not having this property. Assume $\mathcal{S} \neq \emptyset$ and take $I \in \mathcal{S}$ to be a maximal element, which is possible as R is Noetherian. Prime ideals are not in $\mathcal{S}$, so I is not prime, i.e., there exist $a, b \in R$ such that $a, b \notin I$ and $ab \in I$. As I is maximal,

$$I \subsetneq I + (a), I \subsetneq I + (b) \implies I + (a), I + (b) \notin \mathcal{S}$$

Thus there exist P_i, Q_j such that

$$\begin{aligned} &\prod_i P_i \subset I + (a) \qquad && P_i \supset I + (a) \supset I \\ &\prod_j Q_j \subset I + (b) \qquad && Q_j \supset I + (b) \supset I \end{aligned}$$

Now

$$\left(\prod_i P_i\right)\left(\prod_j Q_j\right) \subset (I + (a))(I + (b)) \subset I$$

a contradiction. □

Proof (of Theorem 2.3). Conversely, let M be the unique non-zero prime ideal. Then R is maximal and local. It is now enough to show that M is principal since then every ideal is principal.

Let $y \in M$, $y \neq 0$. Poor man's factorisation gives

$$M^n \subset (y) \subset M$$

for some n. Let n be the smallest such. Then

$$M^n \subset (y), M^{n-1} \not\subset (y)$$

Let $x \in M^{n-1} \setminus (y)$. Set $z = x/y \in \mathrm{Frac}(R)$, $z \notin R$. (At this stage, morally, $z = \pi^{-1}$.) Then

$$xM \subset M^n \subset (y) \implies zM \subset R$$

so zM is an ideal in R. Either $zM = R$, so $M = (z^{-1})R$ and hence M is principal. Or zM is a proper ideal, $zM \subset M$. As R is Noetherian, M is finitely generated. Let $M = (x_1, \ldots, x_n)$, say. Multiplying by z,

$$\begin{pmatrix} zx_1 \\ \vdots \\ zx_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}a_n \end{pmatrix}$$

with $a_{ij} \in R$, i.e.,

$$A(x_i) = z(x_i)$$

working over $\mathrm{Frac}(R)$. This means $\det(A - zI) = 0$. Note $\det(A - zI)$ is a monic polynomial in z with coefficients in R. Since R is integrally closed, $z \in R$. □

Exercise 2. Show that if M is principal then every ideal is principal.

Example. (i) $R = \{m/n \in \mathbb{Q} : (n, p) = 1\}$;
(ii) $R = \{f/q \in k(t) : g(a) \neq 0\}$.

Example. (i) Let $R = \mathbb{Z}$, or $R = \mathcal{O}_K$ where K is a number field. This is Noetherian and integrally closed, but has many non-zero prime ideals.
(ii) Let $v\colon K^* \to \mathbb{R}$ be a non-discrete valuation. Then $\mathcal{O}_K$ is integrally closed and has a unique non-zero prime ideal, but is not Noetherian.
(iii) Let $v\colon \mathbb{Q}^* \to \mathbb{Z}$ be the 2-adic valuation and $\mathcal{O}_v$ be its valuation ring. Then $R = \mathcal{O}_v[2i]$ is local, Noetherian and has a unique non-zero prime ideal, but is not integrally closed.

Chapter 3

Completion

Suppose $K, |\cdot|$ is any valued field. Then K is a metric and topological space, with metric

$$d(x,y) = |x-y|$$

and the topology defined by the open balls

$$B_{a,r} = \{x \in K : |x-a| < r\}$$

We say $x_n \to x$ if $|x_n - x| \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} a_n = A$ if $\sum_{n=1}^{N} a_n \to A$ as $N \to \infty$.

Properties of the absolute value imply that if $x_n \to x, y_n \to x$ then $x_n \pm y_n \to x \pm y$, $x_n y_n \to xy$, and $1/x_n \to 1/x$ provided $x \neq 0$. Hence $+, \times \colon K \times K \to K$ and $(\cdot)^{-1} \colon K^* \to K^*$ are continuous maps.

Definition. $\{x_n\}_{n\in\mathbb{N}}$ is a *Cauchy sequence* if $|x_n - x_m| \to 0$ as $n, m \to \infty$, i.e.,

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall m, n > N \quad |x_n - x_m| < \varepsilon$$

K is *complete* with respect to $|\cdot|$ is every Cauchy sequence converges.

Example. $\mathbb{R}, |\cdot|_\infty$ and $\mathbb{C}, |\cdot|_\infty$ are the only two Archimedean complete fields. $\mathbb{Q}, |\cdot|_\infty$ is non complete, $\mathbb{R}, |\cdot|_\infty$ is its completion. $\mathbb{Q}, |\cdot|_p$ is not complete.

Exercise 3. Argue by countability that $\mathbb{Q}, |\cdot|_\infty$ and $\mathbb{Q}, |\cdot|_p$ are not complete since there are uncountably many Cauchy sequences.

The topological completion of K with respect to $|\cdot|$ can be made into a field, called the *completion* $\hat{K}$ of K with respect to $|\cdot|$. This is constructed as follows.

Let C be the set of Cauchy sequences in K, and note this is a ring containing the ideal I of Cauchy sequences tending to 0. Define $\hat{K} = C/I$, which is a ring.

Then $(x_k) \in \hat{K}^*$ is a Cauchy sequence with $x_k \not\to 0$. Thus for some $\varepsilon > 0$ and sufficiently large k, $|x_k| > \varepsilon$. Hence $(1/x_k)$ is a Cauchy sequence in $\hat{K}^*$ and so $\hat{K}$ is a field.

Note there is a natural injection $K \hookrightarrow \hat{K}$ by $x \mapsto (x, \dots)$. We can extend $|\cdot|$ from K to $\hat{K}$ by

$$|(x_n)| = \lim_{n\to\infty} |x_n|$$

3.1 Properties

(i) $\hat{K}$ is a complete valued field, $|\cdot|_{\hat{K}}$ extends $|\cdot|_K$;
(ii) $K \hookrightarrow \hat{K}$ is dense;
(iii) $K = \hat{K}$ if and only if K is complete;
(iv) K is non-Archimedean if and only if $\hat{K}$ is non-Archimedean;
(v) Equivalent absolute values on K give rise to isomorphic completions (as fields and vector spaces);
(vi) If $\phi\colon K \hookrightarrow L$ is an inclusion of valued fields then there exists a unique $\hat{\phi}\colon \hat{K} \hookrightarrow \hat{L}$ extending ϕ, and this is defined by $\hat{\phi}((x_n)_n) = (\phi(x_n))_n$.

Exercise 4. Check the above statements.

Example. Let K, $|\cdot|$ be non-Archimedean. Theorem 1.2 gives

$$\mathbb{Q}, |\cdot|_\infty \hookrightarrow K, |\cdot| \hookrightarrow \mathbb{C}, |\cdot|_\infty$$

Taking completions, $\mathbb{R} \subset \hat{K} \subset \mathbb{C}$ so $\hat{K} \cong \mathbb{R}$ or $\mathbb{C}$. In particular, $\mathbb{R}$ and $\mathbb{C}$ are the only complete Archimedean fields.

Chapter 4

Local Fields

From now on, we assume every absolute value is non-trivial.

Definition. Let $K, |\cdot|$ be a valued field. $K, |\cdot|$ is a *local field* if it is locally compact as a topological space.

We recall the following properties of topological spaces.

(i) A topological space X is *compact* if every open cover of X has a finite subcover.
(ii) A topological space X is *locally compact* if for every open set $U \subset X$ and $x \in U$ there is an open set U_0 such that $x \in U_0 \subset U$ and U_0 has compact closure.
(iii) A metric space X is locally compact if and only if for every $x \in X$ there exists $R > 0$ such that for all $0 < r < R$ the ball $B_r(x)$ is compact.

Lemma 4.1. The following statements are equivalent.

(i) K is local.
(ii) There exists a compact disc $B_{a,\leq r}$.
(iii) All discs $B_{a,\leq r}$ are compact.

Proof. (iii) $\implies$ (i) $\implies$ (ii) is clear.

We now show (ii) $\implies$ (iii). Take a compact disc $B_{a,\leq r}$. The translation $x \mapsto x + a$ is a homomorphism. Thus $B_{0,\leq r}$ is compact, so $B_{0,\leq s}$ is compact for all $0 < s \leq r$ as a closed subset of a compact set. Now $|\cdot|$ is non-trivial so there exists $\alpha \in K$ such that $|\alpha| > 1$. The map $K \to K, x \mapsto \alpha x$ is continuous so $B_{0,\leq |\alpha|^n r}$ is compact for all $n \in \mathbb{N}$. Hence $B_{0,\leq s}$ is compact for all $0 < s$ and again by translation we are done. $\square$

Proposition 4.2. A local field $K, |\cdot|$ is complete.

Proof. Assume not. Pick $x \in \hat{K} \setminus K$ and a sequence $(x_n)_n$ in K with $x_n \to x$.

Let B be any closed disc that contains all x_n for $n \geq N$ and some $N \in \mathbb{N}$. (For example, there exists $N \in \mathbb{N}$ such that $|x_n - x| < 1$ for all $n \geq N$; now take $B = B_{x_N,\leq 2}$.) B is compact by the above lemma.

Let $U_n = \{y \in K : |y - x| > 1/n\}$. These are open sets and give an open cover of B,

$$B = \bigcup_{n \geq 1} (U_n \cap B)$$

but it has no finite subcover. $\square$

Corollary 4.3. $\mathbb{R}$ and $\mathbb{C}$ are the only Archimedean local fields.

Now suppose $K, |\cdot|$ is non-Archimedean. The following are consequences of the condition $|x+y| \leq \max\{|x|, |y|\}$.

(i) If $|y| < |x|$ then $|x \pm y| = |x|$;
(ii) If $x_1 + \cdots + x_n = 0$ then the two largest absolute values are equal.
(iii) $(x_n)_n$ is Cauchy if and only if $x_n - x_{n-1} \to 0$.
(iv) In a complete field, $\sum_{n=1}^{\infty} x_n$ converges if and only if $x_n \to 0$.

4.1 Topological Properties

(i) If $x \in B_{a,<r}$ then $B_{x,<r} = B_{a,<r}$, so every point in the disc is a centre.
(ii) If B, B' are open discs with $B \cap B' \neq \emptyset$ then $B \subset B'$ or $B' \subset B$.
(iii) Every open disc is also closed.

Proof. Let $K = \bigcup_{a \in K} B_{a,<r}$, a disjoint union of open discs. Then one is the complement of the union of all others (on choosing the radius appropriately small), so every one is closed. □

Theorem 4.4. Suppose $K, |\cdot|$ is non-Archimedean and has the corresponding valuation v. Then the following are equivalent.

(i) K is a local field.
(ii) The valuation ring $\mathcal{O} = \mathcal{O}_v$ is compact.
(iii) K is complete, v is discrete and the residue field $k = \mathcal{O}/M$ is finite.

Proof. (i) $\iff$ (ii): $\mathcal{O} = \{x \in K : |x| \leq 1\} = B_{0,\leq 1}$, and now apply the lemma.

(i),(ii) $\implies$ (iii): K is complete by Proposition 4.2. Write

$$\mathcal{O} = \bigcup_{x \in \mathcal{O}} x + M = \bigcup_{x \in \mathcal{O}} B_{x,<1}$$

a disjoint union of open discs. $\mathcal{O}$ is compact so there exists a finite subcover, hence $\mathcal{O}/M$ is finite. Now take $y \in M \setminus \{0\}$ and write

$$\mathcal{O} = \bigcup_{x \in \mathcal{O}} x + y\mathcal{O}$$

This is a finite union, so the valuation is discrete.

(iii) $\implies$ (ii): Exercise. □

Chapter 5

Discrete Valuation Rings and Completions

Lemma 5.1. Let $\mathcal{O}$ be a DVR and v a valuation of it. Let $K = \mathrm{Frac}(\mathcal{O})$, $M = (\pi)$ for a uniformiser π, and $\mathcal{O}/M = k$ the residue field.

Let $A = \{a_i\}$ be any set of representatives of $\mathcal{O}/M$, $a_i \in \mathcal{O}$, say $0 \in A$. Then every $x \in K^*$ can be written as

$$x = \pi^{v(x)} \sum_{n=0}^{\infty} a_n \pi^n$$

with $a_n \in A$ and $a_0 \neq 0$. We say that a_i are the *digits* in the π-adic expansion of x.

Proof. Write $x = \pi^{v(x)} u$ for some unit $u \in \mathcal{O}^\times$. Reducing mod π,

$$\begin{aligned} \mathcal{O}/M &\xrightarrow{\sim} k \\ u &\mapsto \bar{u} \end{aligned}$$

There exists a unique $a_0 \in A$ such that $\bar{a}_0 = \bar{u}$, so $a_0 - u \in M$. Now write $u = a_0 + \pi u_1$ and reduce u_1 mod π. Then there exists a unique $a_1 \in A$ such that $\bar{a}_1 = \bar{u}_1$. Now write $u = a_0 + \pi a_1 + \pi^2 u_2$ and proceed. We obtain partial sums

$$S_N = \sum_{n=0}^{N} a_n \pi^n \to u$$

in the topology defined by v, because $v(S_n - u) \geq N$ implies $S_N \to u$. Clearly the a_n are unique. □

Remark.
(i) The open balls in K are of the form $x + \pi^n \mathcal{O}$, which is the set of elements of K whose digits coincide with those of x up to a_{n-1}.
(ii) A sequence $(x_k)_k$ in K is Cauchy if and only if the digits of x_k eventually stabilise.
(iii) K is complete with respect to $|\cdot|$ if and only if every Cauchy sequence converges if and only if the inclusion given by the lemma,

$$K \hookrightarrow \left\{ \text{power series } \sum_{n=n_0}^{\infty} a_n \pi^n,\ a_n \in A \right\}$$

is an equality. In general, the RHS is equal to $\hat{K}$.

(iv) $K, \mathcal{O}, M, v$ induce $\hat{K}, \hat{\mathcal{O}}, \hat{M}, \hat{v}$. From the description above, the valuation on $\hat{K}$ is still discrete.

$$\begin{tikzcd} K^* \arrow[r, "v"'] \arrow[d, hook] & \mathbb{Z} \arrow[d, "\iota"] \\ \hat{K}^* \arrow[r, "\hat{v}"'] & \mathbb{Z} \end{tikzcd}$$

Exercise 5. $\mathcal{O}/M^n \to \hat{\mathcal{O}}/\hat{M}^n$ is an isomorphism. In particular, the residue fields are the same and uniformisers stay uniformisers.

Chapter 6

p-adic Numbers

Consider $\mathbb{Q}$ with the p-adic absolute value $|\cdot|_p$ and the p-adic valuation v_p. Then the valuation ring is $\mathcal{O} = \{a/b : p \nmid b\}$, the maximal ideal is $M = (p)$ and the residue field is

$$\mathcal{O}/M \xrightarrow[\text{mod } p]{\sim} \mathbb{F}_p$$

Definition. The *field of p-adic numbers* $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$. The *ring of p-adic integers* $\mathbb{Z}_p$ is its valuation ring. Let $\pi = p$ and $A = \{0, 1, \ldots, p-1\}$ and apply Lemma 5.1.

$$\mathbb{Q} \hookrightarrow \left\{ \sum_{n=n_0}^{\infty} a_n p^n : a_n \in \{0, 1, \ldots, p-1\} \right\} = \mathbb{Q}_p$$

$$\mathbb{Z} \hookrightarrow \mathcal{O} \hookrightarrow \left\{ \sum_{n=0}^{\infty} a_n p^n : a_i \in \{0, 1, \ldots, p-1\} \right\} = \mathbb{Z}_p$$

$$M_{\mathbb{Z}_p} = (p) = \left\{ \sum_{n=1}^{\infty} a_n p^n : a_i \in \{0, 1, \ldots, p-1\} \right\}$$

$$\mathbb{Z}_p / M_{\mathbb{Z}_p} = \mathbb{F}_p$$

Example. Let $p = 3$ and take $A = \{0, 1, 2\}$.

(i) $x = 106 = 1 + 2 \times 3 + 2 \times 3^2 + 3^4$. In general, if $x \in \mathbb{Z}_p$ with $x = \sum_{n=0}^{\infty} a_n p^n$ then $x \in \mathbb{Z}_{\geq 0}$ if and only if this expansion termintates.

(ii) $x = 1/2 \in \mathbb{Z}_3$. Then

$$\frac{1}{2} \mod 3 = \frac{\bar{1}}{\bar{2}} = \bar{2} \in \mathbb{F}_3$$

and this lifts to $2 \in A$, giving the 0th digit.

$$\frac{1}{2} = 2 + \frac{-3}{2} = 2 + 3\frac{-1}{2}$$

Now

$$\frac{-1}{2} \mod 3 = \frac{\overline{-1}}{\bar{2}} = \bar{1} \in \mathbb{F}_3$$

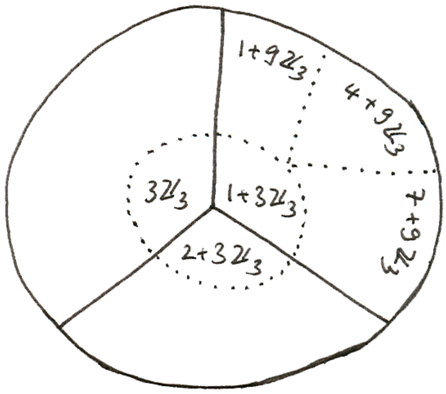
which lifts to $1 \in A$, giving the 1st digit. Continuing this process,

$$\frac{1}{2} = 2 + 1 \cdot 3 + \frac{-9}{2} = 2 + 1 \cdot 3 + 3^2\frac{-1}{2} = 2 + 1 \cdot 3 + 1 \cdot 3^2 + 1 \cdot 3^3 + \cdots$$

Exercise 6. Suppose $x \in \mathbb{Q}_p$. Then $x \in \mathbb{Q}$ if and only if its p-adic expansion is eventually periodic.

Adddition and multiplication work as for decimal expansion, for example,

$$\begin{aligned} \frac{1}{2} &= 2 + 3 + 3^2 + 3^3 + \cdots \\ +\frac{1}{2} &= 2 + 3 + 3^2 + 3^3 + \cdots \\ \frac{1}{2} + \frac{1}{2} &= 4 + 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 + \cdots \\ &= (1+3) + 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 + \cdots = 1 \end{aligned}$$



where the first circle distinguishes the first digit, the second circle distinguishes the first two digits, etc.

$\mathbb{Z}_p$ is topologically homeomorphic to a Cantor set.

6.1 Power Series in $\mathbb{Z}_p$

(i) The geometric series
$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$
converges (in any complete DVR) if and only $|x| < 1$ if and only if $x \in p\mathbb{Z}_p$. For example, in $\mathbb{Z}_3$,
$$\frac{1}{2} + 1 + \frac{1}{1-3} = 1 + (1 + 3 + 3^2 + \cdots) = 2 + 3 + 3^2 + 3^3 + \cdots$$

(ii) The p-adic logarithm has expansion
$$\log_p(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$
It is left as an exercise to show this converges for $x \in \mathbb{Q}_p$ if and only if $|x| < 1$ if and only if $x \in p\mathbb{Z}_p$.

(iii) The exponential function
$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
This converges on $p\mathbb{Z}_p$ for $p > 2$ and on $4\mathbb{Z}_2$ for $p = 2$.

6.2 Additive Structure of $\mathbb{Z}_p$

$\mathbb{Z}_p \subset \mathbb{Q}_p$ is a subgroup of a field of characteristic 0, hence it is a torsion-free abelian group.

We have the following filtration by open and closed subgroups

$$\mathbb{Z}_p \supset p\mathbb{Z}_p \supset p^2\mathbb{Z}_p \supset \cdots$$

where

$$\begin{aligned} p^n\mathbb{Z}_p/p^{n+1}\mathbb{Z}_p &\cong \mathbb{Z}/p\mathbb{Z} \\ x &\mapsto n\text{th } p\text{-adic digit} \end{aligned}$$

6.3 Multiplicative Structure

We know

$$x = \sum_{n=0}^{\infty} a_n p^n$$

is a unit, i.e., in $\mathbb{Z}_p^*$ if and only if $a_0 \neq 0$. $\mathbb{Z}_p^*$ the multiplicative group of units. We have a filtration of subsets

$$\mathbb{Z}_p^* \supset 1 + p\mathbb{Z}_p \supset 1 + p^2\mathbb{Z}_p \supset \cdots$$

and upon writing $U_0 = \mathbb{Z}_p^*$, $U_1 = 1 + p\mathbb{Z}_p$, $U_2 = 1 + p^2\mathbb{Z}_p$, etc. we have, for $n \geq 1$,

$$U_n = \ker\bigl(\mathbb{Z}_p^* \to (\mathbb{Z}/p^n\mathbb{Z})^*, x \mapsto x \mod p^n\bigr)$$

hence this is a subgroup.

$$\begin{aligned} U_0/U_1 &\xrightarrow[\text{mod } p]{\sim} (\mathbb{Z}/p\mathbb{Z})^* && n = 0 \\ U_n/U_{n+1} &\xrightarrow{\sim} (\mathbb{Z}/p\mathbb{Z}, +) && n \geq 1 \end{aligned}$$

These are even isomorphisms of topological groups, under the convention that finite groups are equipped with the discrete topology.

Theorem 6.1. The following are isomorphisms of topological groups

$$\begin{aligned} \mathbb{Z}_p^* &\cong U_1 \times (\mathbb{Z}/p\mathbb{Z})^* \cong \mathbb{Z}_p \times (\mathbb{Z}/p\mathbb{Z})^* && p > 2 \\ \mathbb{Z}_p^* &\cong U_2 \times (\mathbb{Z}/4\mathbb{Z})^* \cong \mathbb{Z}_p \times \{\pm 1\} && p = 2 \end{aligned}$$

Proof. In the cases $p > 2$ and $p = 2$, respectively, consider the maps

$$\begin{aligned} &\exp\colon p\mathbb{Z}_p \to 1 + p\mathbb{Z}_p = U_1 && \log\colon 1 + p\mathbb{Z}_p \to p\mathbb{Z}_p \quad p > 2 \\ &\exp\colon p^2\mathbb{Z}_2 \to 1 + p^2\mathbb{Z}_p = U_2 && \log\colon 1 + p^2\mathbb{Z}_p \to p^2\mathbb{Z}_p \quad p = 2 \end{aligned}$$

These are continuous homomorphisms of groups, and they are inverses to each other. Thus

$$\begin{aligned} U_1 &\cong \mathbb{Z}_p \quad p > 2 \\ U_2 &\cong \mathbb{Z}_p \quad p = 2 \end{aligned}$$

[To prove that the these are indeed inverses and homomorphisms, we need to check that

$$\exp(\log(1+z)) = 1+z$$
$$\log(\exp(z)) = z$$
$$\exp(z+w) = \exp(z)\exp(w)$$

formally as power series, e.g.,

$$\exp(\log(1+z)) = \sum_{m\geq 0} \frac{1}{m!}\left(\sum_{n\geq 1} \frac{(-z)^n}{n}\right)^m = 1+z$$

This is true on $\mathbb{R}$, and now we can use uniqueness of Taylor series.]

Also

$$\begin{aligned} &\mathbb{Z}_p^*/U_1 \cong (\mathbb{Z}/p\mathbb{Z})^* && p>2 \\ &\mathbb{Z}_p^*/U_2 \cong (\mathbb{Z}/p^2\mathbb{Z})^* = (\mathbb{Z}/4\mathbb{Z})^* \cong \{\pm 1\} && p=2 \end{aligned}$$

That is, we have an exact sequence of groups

$$0 \to \mathbb{Z}_p \to \mathbb{Z}_p^* \to (\mathbb{Z}/p\mathbb{Z})^* \to 0$$

and we want to show that is *splits*.

[The sequence

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

of abelian groups is *exact* if $A \xrightarrow{\alpha} B$ is injective and

$$B/A \xrightarrow[\beta]{\sim} C$$

An exact sequence *splits* if there exists a map $\gamma\colon C \to B$ such that $\beta \circ \gamma = \iota$, or equivalently, if $A \times C \xrightarrow{\sim} B$ via $(a, c) \mapsto \alpha(a) + \gamma(c)$.

For example,

$$0 \to \mathbb{Z} \xrightarrow{i_1} \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \xrightarrow{p_2} \mathbb{Z}/2\mathbb{Z} \to 0$$

splits but

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}/2\mathbb{Z} \to 0$$

does not split.]

Consider first the case $p = 2$. $\{\pm 1\} \hookrightarrow \mathbb{Z}_p^*$. It is clear that

$$\{\pm 1\} \subset \mathbb{Z}^* \subset \mathbb{Z}_p^*, \{\pm 1\} \hookrightarrow \mathbb{Z}_2^* \xrightarrow{\text{mod } 4} \{\pm 1\}$$

is the identity.

Now assume $p > 2$. We want $(\mathbb{Z}/p\mathbb{Z})^* \hookrightarrow \mathbb{Z}_p^* \hookrightarrow \mathbb{Q}_p^*$, i.e., what we want is the group of $(p-1)$th roots of unity inside $\mathbb{Q}_p$. Let $1 \leq a \leq p-1$. The details of the following argument are left as an exercise.

Suppose $(a^{p^n})_{n\geq 1}$ is a Cauchy sequence. Then, as $\mathbb{Q}_p$ is complete, $a^{p^n} \to x \in \mathbb{Q}_p$; in fact $x \in \mathbb{Z}_p$ as $\mathbb{Z}_p \subset \mathbb{Q}_p$ is closed. From $x \equiv a \pmod p$ and by continuity, we see that there are at least $p-1$ $(p-1)$th roots of unity in $\mathbb{Q}_p$, namely one for each a. Now use that $\mathbb{Q}_p$ is a field to deduce there are precisely $p-1$ of them. They form a group μ_{p-1} under multiplication.

This gives maps

$$(\mathbb{Z}/p\mathbb{Z})^* \xrightarrow{a \mapsto \lim a^{p^n}} \mathbb{Z}_p^* \qquad\qquad \mathbb{Z}_p^* \xrightarrow{\bmod p} (\mathbb{Z}/p\mathbb{Z})^*$$

and the composition is the identity. □

Corollary 6.2.

$$\mathbb{Q}_p^* \cong \mathbb{Z} \times \mathbb{Z}_p^* \cong \begin{cases} \mathbb{Z} \times \mathbb{Z}_p \times (\mathbb{Z}/p\mathbb{Z})^* & p > 2 \\ \mathbb{Z} \times \mathbb{Z}_p \times \{\pm 1\} & p = 2 \end{cases}$$

Corollary 6.3. There are exactly $p-1$ roots of unity in $\mathbb{Q}_p$ for $p > 2$, and 2 in $\mathbb{Q}_2$.

Proof. Note that $\mathbb{Z}$ and $\mathbb{Z}_p$ are torsion-free. □

Corollary 6.4. For $p > 2$,

$$\mathbb{Q}_p^*/\mathbb{Q}_p^{*2} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}_p/2\mathbb{Z}_p \times (\mathbb{Z}/p\mathbb{Z})^*/(\mathbb{Z}/p\mathbb{Z})^{*2} \cong \{1, p, \eta, \eta p\}$$

for a non-residue η, upon writing $\mathbb{Z}/2\mathbb{Z} \cong \{1, p\}$, $\mathbb{Z}_p/2\mathbb{Z}_p \cong \{1\}$ as 2 is a unit in $\mathbb{Z}_p^*$, and $(\mathbb{Z}/p\mathbb{Z})^*/(\mathbb{Z}/p\mathbb{Z})^{*2} \cong \mathbb{Z}/2\mathbb{Z} \cong \{1, \eta\}$.

For $p = 2$,

$$\mathbb{Q}_p^*/\mathbb{Q}_p^{*2} \cong \{\pm 1, \pm 2, \pm 5, \pm 10\}$$

Corollary 6.5. $\mathbb{Q}_p$ has three quadratic extensions $\mathbb{Q}_p(\sqrt{p})$, $\mathbb{Q}_p(\sqrt{\eta})$, and $\mathbb{Q}_p(\sqrt{\eta p})$ for $p > 2$. For $p = 2$, $\mathbb{Q}_p$ has seven quadratic extensions.

Remark. Compare the last corollary with the following. $\mathbb{R}^*/\mathbb{R}^{*2} = \{\pm 1\}$, $\mathbb{R}$ has one quadratic extension $\mathbb{R}(i) = \mathbb{C}$, and $\mathbb{Q}^*/\mathbb{Q}^{*2}$ is infinite.

Note 1. Suppose K is a field with $\mathrm{char}(K) \neq 2$. Then quadratic extensions of K are in one-to-one correspondence with non-trivial elements of K^*/K^{*2} via

$$K(\sqrt{d}) \mapsto d$$
$$K[X]/(X^2 + aX + b) \hookleftarrow a^2 - nb$$

Note 2. Suppose $p \neq 2$. Under the logarithm map,

$$U_1 = 1 + p\mathbb{Z}_p \to p\mathbb{Z}_p$$
$$U_n = 1 + p^n\mathbb{Z}_p \to p^n\mathbb{Z}_p$$

and $p\mathbb{Z}_p \supset p^n\mathbb{Z}_p$ is the unique subgroup of index p^{n-1} as this is true on the LHS.

Corollary 6.6. Suppose $p \neq 2$. Then

$$(\mathbb{Z}/p^n\mathbb{Z})^* \cong \mathbb{Z}_p^*/U_n \cong (\mathbb{Z}/p\mathbb{Z})^* \times (\mathbb{Z}/p^{n-1}\mathbb{Z})$$

where for the last group on the RHS, $U_1/U_n \cong p\mathbb{Z}_p/p^n\mathbb{Z}_p$. The RHS is a product of cyclic groups of coprime order, so $(\mathbb{Z}/p^n\mathbb{Z})^*$ is cyclic, which is important in basic number theory.

Corollary 6.7. Suppose $p = 2$. Then

$$(\mathbb{Z}/2^n\mathbb{Z})^* \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2^{n-2}\mathbb{Z})$$

generated by -1 and 5.

Note 3. Let $u \in \mathbb{Z}_p^*$. The following are equivalent.

(i) u is a square.
(ii) $u \mod p^n$ is a square in $(\mathbb{Z}/p^n\mathbb{Z})^*$ for all $n \geq 1$.
(iii) If $p > 2$, $u \mod p$ is a square in $(\mathbb{Z}/p\mathbb{Z})^*$. Otherwise, if $p = 2$, $u \mod 8$ is a square in $(\mathbb{Z}/8\mathbb{Z})^*$.

Lemma 6.8. Consider the following system of polynomial equations in $\mathbb{Z}$ or $\mathbb{Z}_p$,

$$V\colon \begin{cases} f_1(x_1, \ldots, x_k) = 0 \\ \vdots \\ f_r(x_1, \ldots, x_k) = 0 \end{cases}$$

Then V has p-adic solution $x \in \mathbb{Z}_p^k$ if and only if V has a solution modulo p^n for all $n \geq 1$. In different notation, $V(\mathbb{Z}_p) \neq \emptyset$ if and only if $V(\mathbb{Z}/p^n\mathbb{Z}) \neq \emptyset$ for all $n \geq 1$.

Proof. The 'if' direction is obvious. For the 'only if' direction, take $x^{(n)} \in \mathbb{Z}_p^k$ with $f_i(x^{(n)}) \equiv 0 \pmod{p^n}$ for all $i = 1, \ldots, r$. As $\mathbb{Z}_p^k$ is compact, there exists a convergent subsequence $x^{(n_i)} \to x \in \mathbb{Z}_p^k$, and by continuity $f_i(x) = 0$ for $i = 1, \ldots, r$. $\square$

We now consider the following setting. Let K be complete with respect to the non-Archimedean absolute value $|\cdot|$ and let $\mathcal{O} = \{x \in K : |x| \leq 1\}$ be its valuation ring.

Theorem 6.9 (Hensel's Lemma, Version 1)**.** Let $f(X) \in \mathcal{O}[X]$ be monic and suppose there exists $x_1 \in \mathcal{O}$ such that

$$\begin{aligned} &|f(x_1)| < 1 && (\iff f(x_1) \in M) \\ &|f'(x_1)| = 1 && (\iff f'(x_1) \in \mathcal{O}^\times = \mathcal{O} \setminus M) \end{aligned}$$

Then there exists a unique $x \in \mathcal{O}$ such that $f(x) = 0$ and $|x - x_1| \leq |f(x_1)|$.

Proof. Choose any $\pi \in M \setminus \{0\}$, not necessarily a uniformiser, such that $\pi \mid f(x_1)$ in $\mathcal{O}$. We proceed by induction on n. Given x_n such that $|x_n - x_1| < |f(x_1)|$ and $f(x_n) \equiv 0 \pmod{\pi^n}$, we want a unique $x_{n+1} \equiv x_n \pmod{\pi^n}$ such that $|x_{n+1} - x_1| < |f(x_1)|$ and $f(x_{n+1}) \equiv 0 \pmod{\pi^{n+1}}$. Then, as $(x_n)_n$ is Cauchy, we take $x = \lim_{n\to\infty} x_n$ and by continuity have $f(x) = 0$. Consider

$$\begin{aligned} \mathcal{O}_K[T] \ni f(x_n + \pi^n T) &= \sum_{j=0}^{\deg(f)} \frac{f^{(j)}(x_n)}{j!} \pi^{nj} T^j \\ &\equiv f(x_n) + f'(x_n)\pi^n T \pmod{\pi^{n+1}} \end{aligned}$$

and recall $f(x_n) \equiv 0 \pmod{\pi^n}$ and $f'(x_n)$ is a unit, as $|f'(x_1)| = 1$ and x_1 is close to x_n. In order to force this to be 0 mod π^{n+1} set $T = -f(x_n)/(f'(x_n)\pi^n) \in \mathcal{O}$, and this is a unique choice modulo T. In other words, if we let

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

then $f(x_{n+1}) \equiv 0 \pmod{\pi^{n+1}}$. $\square$

This is essentially the Newton–Raphson method.

Theorem 6.10 (Hensel's Lemma, Version 2)**.** Let $f_1, \dots, f_r \in \mathcal{O}[x_1, \dots, x_d]$, $r \leq d$. Suppose $x_1 \in \mathcal{O}^d$ is such that

$$\begin{aligned} f_i(x_1) &\equiv 0 \pmod{M} \\ |\frac{\partial f_i}{\partial x_j}(x_1) &\mod M| = r \end{aligned}$$

where $\partial f_i / \partial x_j \in M_{r \times d}(k)$, $k = \mathcal{O}/M$, for every i, j. Then there exists $x \in \mathcal{O}^d$, not in general unique, such that $x \equiv x_1 \pmod{M}$ and $f_i(x) = 0$ for all $i = 1, \dots, r$.

Proof. Similar. □

Theorem 6.11 (Hensel's Lemma, Version 3)**.** Suppose $f \in \mathcal{O}[X]$ is monic and $x_1 \in \mathcal{O}$ is such that $|f(x_1)| < |f'(x_1)|^2$. Then there exists a unique $x \in \mathcal{O}$ such that $f(x) = 0$ and $|x - x_1| < |f(x_1)|/|f'(x_1)|$.

Proof. Similar. □

Example (Square Roots of Unity)**.** This gives an alternative proof that for $u \in \mathbb{Z}_p^*$, if $u \mod p \in \mathbb{F}_p^{*2}$ then $u \in \mathbb{Z}_p^{*2}$.

Suppose $p \neq 2$ and let $u \in \mathbb{Z}_p^*$. Suppose

$$u \equiv \bar{u} = y^2 \pmod{p}$$

in $\mathbb{F}_p^*$. Look at $f(x) = x^2 - u$ and reduce modulo $M = (p)$,

$$\bar{f}(x) = x^2 - \bar{u} = (x - y)(x + y)$$

This has roots $y, -y$ in $\mathbb{F}_p$. Lift y to any element $Y \in \mathbb{Z}_p$ such that $Y \mod p = y$. Then

$$\begin{aligned} f(Y) &= Y^2 - u \equiv 0 \pmod{p} \\ f'(Y) &= 2Y \not\equiv 0 \pmod{p} \end{aligned}$$

so as u is a unit, Y is a unit and $2Y$ is a unit. By Hensel's Lemma, there exists $X \in \mathbb{Z}_p$ such that $X^2 = u$ so $f(x) = x^2 - u = (x - X)(x + X)$ factorises over $\mathbb{Z}_p$.

This does not work for $p = 2$ as 2 is not a unit in $\mathbb{Z}_2$, but Hensel's Lemma (Version 3) still applies.

Exercise 7. A solution modulo 8 lifts to a solution in $\mathbb{Z}_2$.

In general, let K be a complete non-Archimedean field with $\mathcal{O}$, M and $k = \mathcal{O}/M$. Suppose $f(X) \in \mathcal{O}[X]$ is monic and $\bar{f} = f \mod M$ is separable, i.e., $\bar{f}$ has no repeated roots in $\bar{k}$, or equivalently, $\gcd(\bar{f}, \bar{f}') = 1$ in $k[X]$. Then, as $\mathcal{O}$ is integrally closed,

$$\begin{aligned} \{\text{roots of } f(X) \text{ in } K\} &= \{\text{roots of } f(X) \text{ in } \mathcal{O}\} \\ &\leftrightarrow \{\text{roots of } \bar{f} \text{ in } k\} \end{aligned}$$

where the two maps are reduction modulo M and Hensel lifting.

Example (($p-1$)th Roots of Unity in $\mathbb{Z}_p^*$)**.** Let $f(X) = X^p - X$ so $\bar{f}(X) = X(X-1)\cdots(X-(p-1))$. By Hensel's Lemma, $X^p - X$ has p distinct roots $[0], [1], \ldots, [p-1]$ and

$$[\cdot]\colon \mathbb{F}_p^* \to \mathbb{Z}_p^*$$

is a group homomorphism, called *Teichmüller lift.*

Example. Consider $p = 5$, $\mathbb{Q}_5$, $\mathbb{Z}_5$, $k = \mathbb{F}_5$. Then

$$\begin{aligned} &[0] = 0 \qquad [1] = 1 \qquad [-1] = 2 \\ &[2] = 2 + 1 \cdot 5 + 2 \cdot 5^2 + 1 \cdot 5^3 + 3 \cdot 5^4 + \cdots \\ &[3] = 3 + 3 \cdot 5 + 2 \cdot 5^2 + 3 \cdot 5^3 + 1 \cdot 5^4 + \cdots \end{aligned}$$

In general, let K be a complete non-Archimedean field with $\mathcal{O}$, M and suppose $k = \mathcal{O}/M$ is finite (or injects into $\bar{F}_p$). Then there exists a unique group homomorphism

$$[\cdot]\colon k^* \to K^*$$

such that $[x] \mod M = x$, called the Teichmüller lift. To see this, apply Hensel's Lemma to $X^{|k|} - X$.

Chapter 7

Local Fields

Suppose K is a local, non-Archimedean field with $\mathcal{O}$, M, $|\cdot|$ and discrete valuation v, uniformiser π, and $\mathcal{O}/M = k \cong \mathbb{F}_{p^n}$.

By considering Teichmüller lifts, $\{[a] : a \in k\}$ is a set of representatives for $\mathcal{O}/M$, so

$$K = \left\{ \sum_{n=n_0}^{\infty} [a_n]\pi^n : a_n \in k \right\}$$

where $[a_n]$ are called *Teichmüller digits.*

Now suppose $\mathrm{char}(K) = q > 0$. Then since $\mathrm{char}(K) = \mathrm{char}(k)$, we have $q = p$. Then $\mathbb{F}_p \hookrightarrow K$ and, by Hensel's Lemma, K contains the roots of $X^{p^n} - X$. Thus K contains $\mathbb{F}_p(\text{roots of } X^{p^n} - X) = \mathbb{F}_{p^n}$, that is,

$$[\cdot]\colon \mathbb{F}_{p^n} \hookrightarrow K$$

in this case is a field inclusion.

Calling $t = \pi$,

$$K = \left\{ \sum_{m=m_0}^{\infty} a_m t^m : a_m \in \mathbb{F}_{p^n} \right\} = \mathbb{F}_{p^n}((t))$$

$$\mathcal{O} = \left\{ \sum_{m=1}^{\infty} a_m t^m : a_m \in \mathbb{F}_{p^n} \right\} = \mathbb{F}_{p^n}[[t]]$$

So we have a unique local field of positive characteristic, with given residue field $\mathbb{F}_{p^n}$, namely $\mathbb{F}_{p^n}((t))$.

Theorem 7.1. Suppose K is a local field. Then one of the following three cases applies.

(i) $K \cong \mathbb{R}$ or $\mathbb{C}$ (Archimedean);
(ii) $K \cong \mathbb{F}_q((t))$ for a unique $q = p^n$ (Equal Characteristic);
(iii) $[K : \mathbb{Q}_p] < \infty$ for a unique p (Mixed Characteristic).

Proof. We have already seen the two cases K is Archimedean and K is non-Archimedean with $\mathrm{char}(K) > 0$.

So suppose K is local, non-Archimedean and $\mathrm{char}(K) = 0$. Then $\mathbb{Q} \hookrightarrow K$, and so $|\cdot|$ restricts to $\mathbb{Q} \subset K$. Note $|\cdot|_{\mathbb{Q}}$ is non-Archimedean and non-trivial, for otherwise $\mathbb{Z} \hookrightarrow \mathcal{O}$

is an infinite discrete set, hence closed, contradicting that $\mathcal{O}$ is compact. By Ostrowski's Theorem, $|\cdot|_{\mathbb{Q}}$ is $|\cdot|_p$ for some p, so

$$\mathbb{Q}, |\cdot|_p \hookrightarrow K, |\cdot|$$

Taking completions, $\mathbb{Q}_p \hookrightarrow K$, and it suffices to show $[K : \mathbb{Q}_p] < \infty$. Let $k = \mathcal{O}/M$, $k \supset \mathbb{F}_p$, and call $[k : \mathbb{F}_p] = f$. Say k has $\mathbb{F}_p$-basis $\bar{v}_1, \ldots, \bar{v}_f$ and lift these to arbitrary $v_1, \ldots, v_f \in \mathcal{O}$.

The valuation is discrete, and we may assume it is normalised so $v(\pi) = 1$.

$$\begin{tikzcd} K^* \arrow[r, two heads, "v"] & \mathbb{Z} \\ \mathbb{Q}_p^* \ni p \arrow[u, hook] \arrow[r, mapsto] & e > 0 \end{tikzcd}$$

for some integer $e > 0$, because $p \in M_{\mathbb{Q}_p} \subset M_K$ so $v(p) > 0$.

We claim that $[K : \mathbb{Q}_p] \leq ef$ (in fact, it is equal to ef), and $v_i \pi^j$ generate K over $\mathbb{Q}_p$.

Multiplying by a suitable power of p, it is enough to show that, for every $x \in \mathcal{O}$,

$$x = \sum_{\substack{1 \leq i \leq f \\ 1 \leq j \leq e}} A_{ij} v_i \pi^j$$

for some $A_{ij} \in \mathbb{Z}_p$. Clearly,

$$x = \sum_{\substack{1 \leq i \leq f \\ 1 \leq j \leq e}} a_{ij} v_i \pi^j + p \cdot \mathcal{O}$$

for unique $a_{ij} \in \{0, 1, \ldots, p-1\}$, and

$$= \sum_{\substack{1 \leq i \leq f \\ 1 \leq j \leq e}} a_{ij} v_i \pi^j + p \sum_{\substack{1 \leq i \leq f \\ 1 \leq j \leq e}} a'_{ij} v_i \pi^j + \cdots$$

which is a combination of $v_i \pi^j$ with $\mathbb{Z}_p$-coeffients. $\square$

We know that if K is local, non-Archimedean and $\operatorname{char}(K) = 0$ then

$$[K : \mathbb{Q}_p] < \infty$$

Conversely, we will show that every finite extension K of $\mathbb{Q}_p$ has a *unique* structure as a local field, i.e., $|\cdot|_p$ extends to a unique absolute value on K, and K is complete with respect to it.

Chapter 8

Inverse Limits

Instead of considering

$$\mathbb{Q}, |\cdot|_p \xrightarrow{\widehat{\ }} \mathbb{Q}_p \xrightarrow{\text{valutation ring}} \mathbb{Z}_p$$

we consider, given $\mathbb{Z}$ and (p) the quotients $\mathbb{Z}/p^i\mathbb{Z}$ for $i \in \mathbb{N}$ and the projections $\mathbb{Z}/p^i\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^j\mathbb{Z}$ for $j \leq i$.

We call I a *directed set* if I is a set equipped with a partial order $\leq$. Let $(A_i)_{i\in I}$ be a sequence of groups (resp. rings). Let $\pi_{ij} : A_i \to A_j$ be group homomorphism (resp. ring homomorphisms) for $j \leq i$ such that $\pi_{ii} = \iota$ and $\pi_{jk} \circ \pi_{ij} = \pi_{ik}$ for $k \leq j \leq i$.

Definition. The inverse limit is defined as

$$A = \varprojlim_{i\in I}(A_i, \pi_{ij}) = \varprojlim_{i\in I} A_i = \{(a_i)_{i\in I} : \pi_{ij}(a_i) = a_j \ \forall j \leq i\}$$

This is a group (resp. ring).

Example. $\mathbb{N}$ can be made a directed set via $(\mathbb{N}, \leq)$ or $(\mathbb{N}, |)$.

Example. Let $I = (\mathbb{N}, \leq)$, $(A_i)_{i\in\mathbb{N}} = (\mathbb{Z}/p^i\mathbb{Z})_{i\in\mathbb{N}}$ for some prime p, and let

$$\pi_{ij} \colon \mathbb{Z}/p^i\mathbb{Z} \to \mathbb{Z}/p^j\mathbb{Z}$$

for $j \leq i$. Then

$$\mathbb{Z}_p = \varprojlim_{i\in\mathbb{N}} \mathbb{Z}/p^i\mathbb{Z}$$

Similarly,

$$k[[t]] = \varprojlim_{i\in\mathbb{N}} k[t]/(t^i)$$

where π_{ij} is truncation modulo t^j.

Example. Let $I = (\mathbb{N}, |)$, $A_i = \mathbb{Z}/i\mathbb{Z}$, and

$$\pi_{ij} \colon \mathbb{Z}/i\mathbb{Z} \xrightarrow{\text{mod } j} \mathbb{Z}/j\mathbb{Z}$$

for $j \mid i$. Then

$$\varprojlim_{i\in\mathbb{N}} \mathbb{Z}/i\mathbb{Z} = \hat{\mathbb{Z}}$$

is a ring. It is left as an exercise to also show

$$\hat{\mathbb{Z}} = \prod_{p \text{ prime}} \mathbb{Z}_p$$

Remark. (i) $A = \varprojlim_{i\in I}(A_i, \pi_{ij})$. This has the universal property that we have projections $A \xrightarrow{p_i} A_i$ such that $p_i\pi_{ij} = p_j$. For any B with such maps $B \xrightarrow{q_i} A_i$, $q_i\pi_{ij} = q_j$ there exists a unique $\phi\colon B \to A$

$$\begin{tikzcd} A \arrow[dr, "p_i"] & \\ B \arrow[u, "\phi"] \arrow[r, "q_i"'] & A_i \end{tikzcd}$$

(ii) If the A_i have a topology, the *inverse limit topology* on $A = \varprojlim_{i\in I} A_i$ is the weakest topology that makes all $A \xrightarrow{p_i} A_i$ continuous.

Example. On $\mathbb{Z}_p$ the inverse limit topology coincides with the p-adic topology. The LHS is generated by preimages of points under $\mathbb{Z}_p \to \mathbb{Z}/p^i\mathbb{Z}$, and the RHS is generated by open balls $a + p^i\mathbb{Z}_p$.

Let $K, |\cdot|$ be non-Archimedean with $\mathcal{O}$, M, and corresponding valuation v. Choose $\pi \in M \setminus \{0\}$ and normalise such that $v(\pi) = 1$.

Proposition 8.1. *K is complete with respect to $|\cdot|$ if and only if*

$$\mathcal{O} \to \varprojlim_{i\in\mathbb{N}} \mathcal{O}/\pi^i\mathcal{O}, x \mapsto (x \mod \pi^i)_{i\in\mathbb{N}} \tag{$*$}$$

is an isomorphism.

Note 4. $\varprojlim_{i\in\mathbb{N}} \mathcal{O}/\pi^i\mathcal{O}$ is the completion of the ring $\mathcal{O}$ with respect to the ideal (π).

Note 5. If $(*)$ is an isomorphism, we say $\mathcal{O}$ is (π)-adically complete (cf. surjective) and separated (cf. injective).

Proof. Write $K = \bigcup_x x + \mathcal{O}$ as a disjoint union of open and closed sets, where x runs over coset representatives of $\mathcal{O}$ in K. Note that K is complete with respect to $|\cdot|$ if and only if each $x+\mathcal{O}$ is complete with respect to $|\cdot|$ if and only if $\mathcal{O}$ is complete with respect to $|\cdot|$. Equivalently,

$$\forall (x_n)_n \text{ in } \mathcal{O} \text{ with } |x_n - x_{n+1}| \to 0 \quad \exists x \in \mathcal{O} \quad |x_n - x| \to 0 \tag{A}$$

and such an x is necessarily unique.

Now $(*)$ is an isomorphism if and only if

$$\forall (x_i)_i \text{ in } \mathcal{O} \text{ with } v(x_{i+1} - x_i) \geq i \quad \exists! x \in \mathcal{O} \quad v(x - x_i) \geq i \tag{B}$$

But every sequence in (A) has a subsequence as in (B), so they are equivalent. □

8.1 Profinite Groups

Let I be a directed set and $\pi_{ij}\colon G_i \to G_j$ group homomorphisms. Suppose $(G_i)_{i\in I}$ be a directed system of finite groups, all with the discrete topology. Then

$$G = \varprojlim_{i\in I} G_i$$

is a *profinite group* with the inverse limit topology, called the profinite topology.

Note 6. If the G_i are compact (e.g. finite) then $G \subset \prod_{i \in I} G_i$ is compact and so G is compact in the profinite topology.

Example. $(\mathbb{Z}_p, +)$, $(\mathbb{Z}_p^*, \times)$, $(\mathbb{F}_{p^n}[[t]], +)$, $(\hat{\mathbb{Z}}, +)$.

8.2 Main Example

Let L/K be an extension of fields, possibly infinite, but the union of finite Galois extensions k/K. For example, $\bar{\mathbb{Q}}/\mathbb{Q}$, $\bar{\mathbb{F}}_p/\mathbb{F}_p$, recalling $\bar{\mathbb{F}}_p = \bigcup_{n \geq 1} \mathbb{F}_{p^n}$. Then

$$G = \mathrm{Gal}(L/K) = \mathrm{Aut}(L/K) = \varprojlim_k \mathrm{Gal}(k/K)$$

is a profinite group.

By the Fundamental Theorem of Galois Theory, extensions of K in L are in one-to-one correspondence with closed subgroups of G, and finite extensions of K in L are in one-to-one correspondence with open subgroups of G.

Exercise 8. Show the following

$$\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \cong (\hat{Z}, +)$$
$$\mathrm{Frob}_q \colon x \mapsto x^q \leftrightarrow 1$$
$$\mathrm{Gal}(\mathbb{Q}\Big(\bigcup_{n \geq 1} \mu_{p^n}\Big)/\mathbb{Q}) \cong \mathbb{Z}_p^*$$
$$\mathrm{Gal}(\mathbb{Q}\Big(\bigcup_{n \geq 1} \mu_n\Big)/\mathbb{Q}) \cong \hat{Z}^*$$

Note $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is presently still unknown.

Chapter 9

Extensions of Complete Fields

Example.

$$\begin{tikzcd}
\mathbb{Z}[i] & P \arrow[l, hook'] \arrow[d] & 1+i \arrow[d, dash] & 3 \arrow[d, dash] & 2+i \arrow[d, dash] & 2-i \arrow[dl, dash] & 7 \arrow[d, dash] \\
\mathbb{Z} \arrow[u, hook] & P \cap \mathbb{Z} & 2 & 3 & 5 & & 7
\end{tikzcd}$$

$|\cdot|_5$ on $\mathbb{Q}$ has two extensions to $\mathbb{Q}(i)$, $|\cdot|_{2+i}$ and $|\cdot|_{2-i}$. This cannot happen for complete fields. “Primes do not split”.

Theorem 9.1. Suppose $K, |\cdot|$ is complete non-Archimedean and L/K is finite with $[L : K] = d$. Then

(i) There is a unique absolute value $|\cdot|_L$ on L extending $|\cdot|$ on K, and L is complete with respect to this.
(ii) $|x|_L = |N_{L/K}(x)|^{1/d}$.
(iii) $\mathcal{O}_L$ is the integral closure of $\mathcal{O}_K$ in L.

Proof. **Uniqueness.** We do more. Given $K, |\cdot|$ and a vector space V over K, a *norm* on V is a map $\|\cdot\| \colon V \to \mathbb{R}$ such that

- $\|v\| = 0$ if and only if $v = 0$;
- $\|\alpha v\| = |\alpha| \|v\|$;
- $\|v + w\| \leq \max\{\|v\|, \|w\|\}$.

For example, $V = K^d$, $\|v\|_{\text{sup}} = \max_i |v_i|$ for $v = (v_1, \ldots, v_d)$. We say two norms are *equivalent*, denoted $\|\cdot\|_1 \sim \|\cdot\|_2$, if there exists $c, C \in \mathbb{R}_{>0}$ such that

$$c\|\cdot\|_1 \leq \|\cdot\|_2 \leq C\|\cdot\|_1$$

A norm defines a metric on V, and equivalent norms induce the same topology.

Proposition 9.2. Let $K, |\cdot|$ be complete non-Archimedean and V be a finite-dimensional K-vector space. Then any two norms on V are equivalent, and V is complete with respect to any of them.

Proof. By induction on $d = \dim V$. If $d = 1$ then $V = K \cdot e$, $\|\alpha e\| = |\alpha| \|e\|$, so any two norms on V are multiples of each other.

Suppose $d > 1$, let $V = K^d$ with norm $\|\cdot\|$. We show that $\|\cdot\| \sim \|\cdot\|_{\text{sup}}$. This also proves that V is complete. Let $e_1, \dots, e_d$ be a basis of V, let $C = \max_i \|e_i\|$. Then

$$v = \sum_{i=1}^{d} v_i e_i \implies \|v\| \le \max_i \|v_i e_i\| \le C\|v\|_{\text{sup}}$$

It suffices to prove $c\|v\|_{\text{sup}} \le \|v\|$ for some $0 < c$. Suppose not and take a sequence $v^{(n)}$ in $V \setminus \{0\}$ such that

$$\frac{\|v^{(n)}\|}{\|v^{(n)}\|_{\text{sup}}} \to 0$$

as $n \to \infty$. Then, for some $i \in \{1, \dots, d\}$,

$$\|v^{(n)}\|_{\text{sup}} = |v_i^{(n)}|$$

for infinitely many $n \in \mathbb{N}$. Without loss of generality assume $i = d$, replace $v^{(n)}$ by this subsequence and rescale

$$v^{(n)} \mapsto \frac{1}{v_d^{(n)}} v^{(n)}$$

so that

(i) $v_d^{(n)} = 1$ for all $n \in \mathbb{N}$;
(ii) $v^{(n)}$ is in $\mathcal{O}_K^d \subset K^d$, i.e., $\|v^{(n)}\|_{\text{sup}} = 1$;
(iii) $\|v^{(n)}\| \to 0$ as $n \to \infty$.

Let $u^{(n)} = v^{(n)} - e_d \in \mathcal{O}_k^{d-1} \subset \mathcal{O}_K^d$. Then

$$\|u^{(n+1)} - u^{(n)}\| = \|v^{(n+1)} - v^{(n)}\| \le \max\{\|v^{(n+1)}\|, \|v^{(n)}\|\} \to 0$$

as $n \to \infty$. By induction, $(K^{d-1}, \|\cdot\|)$ is complete, so

$$u^{(n)} \to u \in K^{d-1}$$

and

$$\|e_d + u\| = \lim_{n\to\infty} \|e_d + u^{(n)}\| = \lim \|v^{(n)}\| = 0$$

so $e_d + u = 0$ but $u \in K^{d-1}$, $e_d \notin K^{d-1}$, contradiction. □

Now suppose $|\cdot|_1$, $|\cdot|_2$ are absolute values on L extending $|\cdot|$ on K. They are norms on L, hence equivalent, i.e.,

$$c|x|_1 \le |x|_2 \le C|x|_1$$

But $|x^n|_1 = |x|_1^n$ and $|x^n|_2 = |x|_2^n$, so

$$c^{1/n}|x|_1 \le |x|_2 \le C^{1/n}|x|_1$$

As $c^{1/n}, C^{1/n} \to 1$ as $n \to \infty$,

$$|x|_1 = |x|_2$$

Existence. Let $|x|_L = |N_{L/K}(x)|^{1/d}$. We want to prove this is an absolute value on L. Clearly

(i) $|x|_L$ extends $|\cdot|$ on K;

(ii) $|x|_L = 0$ if and only if $x = 0$;
(iii) $|xy|_L = |x|_L |y|_L$.

It suffices to prove $|x+y|_L \leq \max\{|x|_L, |y|_L\}$. Without loss of generality $|y|_L \leq |x|_L$, so $z = y/x$ has $|z|_L \leq 1$, and

$$|x+y|_L \leq |x|_L \iff |1+z|_L \leq 1$$

so we claim

$$N_{L/K}(z) \in \mathcal{O}_K \implies N_{L/K}(1+z) \in \mathcal{O}_K$$

Let f be the minimal polynomial of z over K. Then

$$N_{L/K}(z) = \pm f(0)^m$$

where $m = [L : K]/\deg(f)$. Then, as f is monic irreducible in $K[X]$ and $f(0)^m \in \mathcal{O}_K$ hence $f(0) \in \mathcal{O}$ (as $\mathcal{O}_K$ is integrally closed), we deduce $f(z) \in \mathcal{O}_K[z]$ (see Question 23). Therefore,

$$N_{L/K}(1+z) \in \mathcal{O}_K$$

Finally, $\mathcal{O}_L$ is the integral closure of $\mathcal{O}_K$ in L, because we know that $N_{L/K}(z) \in \mathcal{O}_K$ if and only if z is integral over K. □

9.1 Consequences

Suppose $K, |\cdot|$ is complete non-Archimedean. Then $|\cdot|$ extends uniquely to $\bar{K}$,

$$\alpha \in \bar{K} \implies |\alpha| = |N_{K(\alpha)/K}(\alpha)|^{1/[K(\alpha):K]}$$

Note that, in general, $\bar{K}$ is not complete, e.g., $\bar{\mathbb{Q}}_p$ is not. Its completion is $\mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p$, which is complete and algebraically closed.

If $\alpha, \alpha' \in \bar{K}$ are Galois conjugates over K then $|\alpha| = |\alpha'|$. To see this, note that α, α' are Galois conjugates if and only if α, α' are roots of the same irreducible polynomials $f \in K[X]$, and then α, α' have the same norm.

Lemma 9.3 (Krasner's Lemma)**.** Let $f(X) \in K[X]$ be irreducible and monic, say

$$f(x) = \prod_{i=1}^{d}(X - \alpha_i)$$

over $\bar{K}$. Suppose $\beta \in \bar{K}$ is such that $|\beta - \alpha_1| < |\beta - \alpha_i|$ for all $i > 1$. Then $\alpha_1 \in K(\beta)$.

Proof. $\alpha_1 \neq \alpha_2, \ldots, \alpha_d$, so α_1 is a simple root of f, so f is separable. Let $L = K(\beta)$, and note $L' = L(\alpha_1, \ldots, \alpha_d)$ is a Galois extension of L. For $\sigma \in \mathrm{Gal}(L'/L)$,

$$|\beta - \sigma(\alpha_1)| = |\sigma(\beta - \alpha_1)| = |\beta - \alpha_1| \neq |\beta - \alpha_i|$$

for all $i > 1$. Thus $\sigma(\alpha_1) \neq \alpha_2, \ldots, \alpha_d$, so $\sigma(\alpha_1) = \alpha_1$. Hence the minimal polynomial of α_1 over L has degree 1, so $\alpha_1 \in K(\beta)$. □

As a consequence of this, $\mathbb{Q}_p$ has only finitely many extensions of a given degree.

Now suppose L/K is finite. We have

$$\begin{aligned} K, |\cdot| &\hookrightarrow L, |\cdot| \\ \mathcal{O}_K &\hookrightarrow \mathcal{O}_L \\ M_K &\hookrightarrow M_L \\ k_K &\hookrightarrow k_L \end{aligned}$$

Proposition 9.4. Suppose $[L : K] = d$, L/K is separable and the valuations are discrete. Then

$$\mathcal{O}_L \cong \mathcal{O}_K^d$$

as $\mathcal{O}_K$-modules.

Remark. All conditions are necessary, and the result does not hold for general number fields.

Proof. We want to show that $\mathcal{O}_L$ injects into a free $\mathcal{O}_K$-module of finite rank. Then, as $\mathcal{O}_K$ is a PID, $\mathcal{O}_L$ is free of finite rank, and since $\mathrm{Frac}(\mathcal{O}_K) = K$, $\mathrm{Frac}(\mathcal{O}_L) = L$ we conclude $\mathcal{O}_L$ has rank $[L : K] = d$.

Let $B(x, y) = \mathrm{Tr}_{L/K}(xy)$, $L \times L \to K$. It is a non-degenerate K-bilinear form, where the non-degeneracy of this form is equivalent to the separability of L/K.

Pick $e_1, \dots, e_d \in \mathcal{O}_L$ a basis for L/K, and let $e_1^\checkmark, \dots, e_d^\checkmark \in L$ be the dual basis with respect to B, i.e., $B(e_i, e_j^\checkmark) = \delta_{ij}$. Then

$$\mathcal{O}_L \subset \{x \in L : B(x, y) \in \mathcal{O}_K \ \forall y \in \mathcal{O}_L\} =: \mathcal{O}_L^\checkmark$$

as $\mathrm{Tr}_{L/K}(\mathcal{O}_L) \subset \mathcal{O}_K$, because $\mathcal{O}_L/\mathcal{O}_K$ is integral. Further,

$$\mathcal{O}_L^\checkmark \subset \bigoplus_{i=1}^{d} \mathcal{O}_k \cdot e_i^\checkmark$$

a finitely generated free $\mathcal{O}_K$-module. $\square$

Chapter 10

Ramification and Inertia

Consider the following setting. Suppose we have K, $|\cdot|$, v_k, $\mathcal{O}_K$, M_K, k_K and L, $|\cdot|$, v_L, $\mathcal{O}_L$, M_L, k_L, where $[L:K]=d$, all with their usual meaning.

Suppose v_K is discrete and normalised. Assume k_K is *perfect*, i.e., every finite extension of k_K is separable. This is the case, e.g., for $\mathbb{F}_q$, $\bar{\mathbb{F}}_p$, fields of characteristic 0, or algebraically closed fields. It is not true for $\mathbb{F}_p(t)$.

Definition. The *residue degree*, or *inertial degree*, is $f = f_{L/K} = [k_L : k_K]$.

Let the valuations on K^* and L^* be given by $\log_\alpha |\cdot| \colon K^* \twoheadrightarrow \mathbb{Z}$ and $\log_\alpha |N_{L/K}(\cdot)|^{1/d} \colon L^* \to \mathbb{R}$ with image in $(1/d)\mathbb{Z}$, respectively. Then the image is a discrete subgroup, so v_L is discrete.

Definition. The index $[v_L(L^*) : v_L(K^*)]$ is called the *ramification index* of L/K, denoted $e = e_{L/K}$. Equivalently, $(\pi_K) = (\pi_L^e)$ in $\mathcal{O}_L$.

Remark. There are two possible conventions.

(i) Normalise $v_L, |\cdot|_L$, but then $v_L|_{K^*} \neq v_K$ but $v_L|_{K^*} = e v_K$.
(ii) Let $v_L \colon L^* \to (1/e)\mathbb{Z}$, then v_L is not normalised but $v_L|_{K^*} = v_K$.

Both are used.

Proposition 10.1.
$$e_{L/K} f_{L/K} = [L:K]$$

Proof. We know $\mathcal{O}_L \cong \mathcal{O}_K^d$ as $\mathcal{O}_K$-modules where $d = [L:K]$. Thus
$$\mathcal{O}_L/\pi_K \mathcal{O}_L \cong \mathcal{O}_K^d / \pi_K \mathcal{O}_K^d \cong (\mathcal{O}_K/\pi_K \mathcal{O})^d \cong k_K^d$$
as $\mathcal{O}_K$-modules. But
$$\mathcal{O}_L \supset \pi_L \mathcal{O}_L \supset \pi_L^2 \mathcal{O}_L \supset \cdots \supset \pi_L^e \mathcal{O}_L = \pi_K \mathcal{O}_L$$
and
$$\pi_L^i \mathcal{O}_L / \pi_L^{i+1} \mathcal{O}_L \cong \mathcal{O}_L / \pi_L \mathcal{O}_L, \quad x \mapsto x \mod \pi_L^i$$
as $\mathcal{O}_K$-modules, so these are also isomorphic to $k_L \cong k_K^f$ as $\mathcal{O}_K$-modules. Comparing dimensions, $d = ef$. □

Definition. L/K is *unramified* if $e_{L/K} = 1$, or equivalently, $[L:K] = f_{L/K}$. L/K is *totally ramified* if $e_{L/K} = [L:K]$, or equivalently, $f_{L/K} = 1$, that is, $k_L \cong k_K$.

Note 7. Given extensions $M/L/K$,

$$f_{M/K} = f_{M/L} f_{L/K}$$

by the tower law for $k_M/k_L/k_K$, and

$$e_{M/K} = e_{M/L} e_{L/K}$$

by the tower law for $M/L/K$.

Example. Suppose $[L : \mathbb{Q}_p] = 2$ for some $p \neq 2$. Then

$$L = \mathbb{Q}_p(\sqrt{d})$$

for some $d \in \{p, \eta p, \eta\}$ with $\bar{\eta} = \eta \mod p \in \mathbb{F}_p^*$ a non-residue.

- $L = \mathbb{Q}_p(\sqrt{p})$, $\mathbb{Q}_p(\sqrt{\eta p})$ have $e_{L/\mathbb{Q}_p} \geq 2$ as $p = \sqrt{\eta p}^2 u$ for some unit u, so $v(\sqrt{\eta p}) = 1/2$ in L. So $e = 2$, $f = 1$, and this is a totally ramified extension.
- $L = \mathbb{Q}_p(\sqrt{\eta})$ has $f_{L/\mathbb{Q}_p} \geq 2$. L contains the roots of $X^2 - \eta$, so

$$k_L \supset \mathbb{F}_p(\text{roots of } X^2 - \bar{\eta}) = \mathbb{F}_{p^2}$$

as $X^2 - \bar{\eta}$ is irreducible over $\mathbb{F}_p$. So $e = 1$, $f = 2$ and this is an unramified extension.

Exercise 9. For $L/\mathbb{Q}_2$, the extension $L = \mathbb{Q}_2(\mu_3) = \mathbb{Q}_2(\sqrt{-3}) = \mathbb{Q}_2(\sqrt{5})$ is unramified, but the other six quadratic extensions are ramified.

Chapter 11

Unramified Extensions

Note that these are in one-to-one correspondence with extensions of the residue fields.

Theorem 11.1. (i) Suppose L/K is a finite, unramified extension. Then $\mathcal{O}_L = \mathcal{O}_K[\alpha]$ (and so $L = K(\alpha)$) for any $\alpha \in \mathcal{O}_L$ with $k_L = k_K(\bar{\alpha})$.

(ii) Suppose ℓ/k_K is finite. There exists an unramified extension L/K with $k_L \cong \ell$ over k_K.

(iii) Suppose L/K is a finite, unramified extension and let L'/K be any finite extension. Then

$$\mathrm{Hom}_K(L, L') \to \mathrm{Hom}_{k_K}(k_L, k_{L'})$$

is a bijection.

Proof. (i) We know $\mathcal{O}_L \cong \mathcal{O}_K^d$ as $\mathcal{O}_K$-modules, where $d = [L : K]$. As the extension L/K is unramified, $\pi_L = \pi_K$. Now $k_L = k_K(\bar{\alpha})$ implies $1, \bar{\alpha}, \ldots, \bar{\alpha}^{d-1}$ generate $\mathcal{O}_L/\pi_L\mathcal{O}_L$. Therefore, by Nakayama's lemma, as $\mathcal{O}_L$ is local, $1, \alpha, \ldots, \alpha^{d-1}$ generate $\mathcal{O}_L$ as an $\mathcal{O}_K$-module.

(ii) Write $\ell = k_K(\bar{\alpha})$. We can do this since k_K is perfect and so ℓ/k_K is separable, hence there exists a primitive element.
Lift the minimal polynomial $\bar{g}(X) \in k_K[X]$ of $\bar{\alpha}$ to any monic polynomial $g(X) \in \mathcal{O}_K[X]$, so $g(X)$ is irreducible. Now let $L = K(\text{roots of } g) = K[X]/(g(X))$.

(iii) Write $L = K(\alpha)$ as in (ii); so $k_L = k_K(\bar{\alpha})$. Let $g(X)$ be the minimal polynomial of α over K and let $\bar{g}(X)$ be its reduction over k_K.
Given $\tilde{\phi}\colon k_L \to k_{L'}$, we find a root $\tilde{\phi}(\bar{\alpha})$ of $\bar{g}(X)$ in $k_{L'}$. By Hensel's Lemma, there exists a unique root of $g(X)$ in L lifting to it, and hence there exists a unique $\phi\colon L \to L'$ lifting $\tilde{\phi}$.

□

Remark. Part (iii) implies the field L in (ii) is unique up to isomorphism.

Corollary 11.2. Suppose L/K is a finite, unramified extension. Then L/K is Galois if and only if k_L/k_K is Galois, and if this is the case then $\mathrm{Gal}(L/K) \cong \mathrm{Gal}(k_L/k_K)$.

Proof. K-automorphisms $\sigma\colon L \to L$ and k_K-automorphisms $\bar{\sigma}\colon k_L \to k_L$ are in one-to-one correspondence under the maps reduction modulo M_K and its inverse by part (iii) of Theorem 11.1. So

$$\mathrm{Aut}(L/K) \cong \mathrm{Aut}(k_L/k_K)$$

In particular, L/K is Galois if and only if $|\mathrm{Aut}(L/K)| = [L : K]$ if and only if $|\mathrm{Aut}(k_L : k_K)| = [k_L : k_K] = [L : K]$ if and only if k_L/k_K is Galois. □

Example. Suppose $K = \mathbb{Q}_p$. (The situation is similar for any local field K.) $\mathbb{Q}_p$ has a unique ramified extension of degree $n \geq 1$,

$$L_n = \mathbb{Q}_p(\mu_{p^n-1})$$

$L_n/\mathbb{Q}_p$ is Galois with $\mathrm{Gal}(L_n/\mathbb{Q}_p) \cong \mathbb{Z}/n\mathbb{Z}$.

This is because $\mathbb{F}_p$ has a unique extension of degree n,

$$\mathbb{F}_{p^n} = \mathbb{F}_p(\text{roots of } X^{p^n} - X) = \mathbb{F}_p(\mu_{p^n-1})$$

and $\mathbb{F}_{p^n}/\mathbb{F}_p$ is Galois with Galois group $\mathbb{Z}/n\mathbb{Z}$.

Corollary 11.3. Suppose L/K is finite. There exists a unique maximal unramified extension K' of K in L, so

$$\begin{tikzcd} L \\ K' \arrow[u, "\text{totally ramified}"] \\ K \arrow[u, "\text{unramified}"] \end{tikzcd}$$

and every unramified extension of K in L is contained in K'.

Proof.

$$\begin{tikzcd} L & k_L \\ K \arrow[u] & k_K \arrow[u] \end{tikzcd}$$

By part (iii) of Theorem 11.1, there exists an unramified extension K' of K with $k_{K'} \cong k_L$ over k_K. By part (ii) of Theorem 11.1, $K' \hookrightarrow L$ over K.

L/K' is totally ramified, i.e., $f_{L/K'} = 1$, and part (iii) of Theorem 11.1 gives the last claim. ☐

Note 8. If L/K is Galois then K'/K is Galois.

Example. Suppose $p \neq 2$ is a prime and let $K = \mathbb{Q}_p$. Then

$$\begin{tikzcd} L = \mathbb{Q}_p(\sqrt{p}, \sqrt{\eta}) & \\ & \mathbb{Q}_p(\sqrt{\eta}) \arrow[ul, "2"'] \\ \mathbb{Q}_p \arrow[uu, "e=f=2"] \arrow[ur, "2"'] & \end{tikzcd}$$

$\mathbb{Q}_p(\sqrt{\eta})$ is the maximal unramified extension of $\mathbb{Q}_p$ in L.

Chapter 12

Totally Ramified Extensions

Proposition 12.1. Suppose L/K is totally ramified of degree e, $\pi = \pi_L$ is a uniformiser and v_L is a normalised valuation, i.e., $v_L(\pi) = 1$. Then

(i) π satisfies an Eisenstein polynomial of degree e over $\mathcal{O}_K$.
(ii) $\mathcal{O}_L = \mathcal{O}_K[\pi]$.

Conversely, if $g \in \mathcal{O}_K[X]$ is Eisenstein then $L = K[X]/g(X)$ is totally ramified over K and $v_L(\text{root of } g) = 1$.

Recall that $g(X)$ is *Eisenstein* if and only if

$$g(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0$$

with $v(a_i) \geq 1$, $v(a_0) = 1$. Then $g(X)$ is irreducible by Eisenstein's criterion.

Proof. (i) Consider the minimal polynomial of π over K,

$$\pi^n + a_{n-1}\pi^{n-1} + \cdots + a_0 = 0$$

irreducible over K and $n \leq e = [L : K]$. π is integral over $\mathcal{O}_K$, so $a_i \in \mathcal{O}_K$ for all $i = 0, \ldots, n-1$. Now consider the valuation of the LHS. The sum is 0 so two terms have the same smallest valuation.

$$v_L(a_i\pi^i) = i + ev_K(a_i) \equiv i \pmod{e}$$

So these are all distinct for $i < e$. Hence $n = e$, $v_L(a_0) = v_L(\pi^n) = n = e$, in other words, $v_K(a_0) = 1$, and $v_K(a_i) > 0$ because $v_K(a_0)$ and $v_K(a_n)$ are smallest. Thus $g(X)$ is Eisenstein, irreducible, and $L = K(\pi)$.

(ii) For $x \in L$ write $x = \sum_{i=0}^{e-1} b_i\pi^i$ with $b_i \in K$. Then

$$v_L(x) = \min_i\{i + ev_K(b_i)\}$$

as the elements $i + ev_K(b_i)$ all have distinct valuations. Now $x \in \mathcal{O}_L$ if and only if $v_L(x) \geq 0$ if and only if $v_K(b_i) \geq 0$ for all $i = 0, \ldots, e-1$. So $\mathcal{O}_L = \mathcal{O}_K[\pi]$.

Conversely, if $g(X) \in \mathcal{O}_K[X]$ is Eisenstein, let

$$L = K[X]/g(X) \cong K(\text{root of } g) = K(\alpha)$$

for some $\alpha \in L$. Then

$$\alpha \in L \implies \alpha \in \mathcal{O}_L$$

as $\mathcal{O}_L$ is integrally closed, and as $g \mod M = X^n$,

$$\alpha \in M_L$$

so $v_L(\alpha) > 0$. Consider

$$\alpha^n + \underbrace{a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha}_{v_L(\cdot) > e_{L/K}} + \underbrace{a_0}_{v_L(\cdot) = e_{L/K}} = 0$$

so $v_L(\alpha^n) = e_{L/K}$, so $n = [L : K] \mid e_{L/K}$, but $e_{L/K} \mid n$. Thus $n = e_{L/K}$, L/K is totally ramified, and $v_L(\alpha) = 1$. $\square$

Chapter 13

Inertia Group and Higher Ramification

Suppose L/K is Galois, $G = \mathrm{Gal}(L/K)$. Let K' be the maximal unramified extension of K in L and recall K'/K is also Galois.

$$\begin{tikzcd}
L \arrow[d, dash, "\text{totally ramified}"'] \arrow[dd, dash, bend left=60, "G"] \\
K' \arrow[d, dash, "\text{unramified}"'] \\
K
\end{tikzcd}$$

We also have $\mathrm{Gal}(K'/K) = \mathrm{Gal}(k_L/k_K)$.

Definition. The *inertia group* $I = I_{L/K}$ is $\mathrm{Gal}(L/K')$. Equivalently,

$$\begin{aligned} I &= \{\sigma \in G : \sigma \text{ maps to } \iota \text{ in } \mathrm{Gal}(k_L/k_K)\} \\ &= \{\sigma \in G : \sigma x \equiv x \pmod{M}\ \forall x \in \mathcal{O}_L\} \end{aligned}$$

Note 9. $I \triangleleft G_{L/K}$, $e = |I|$.

Example. Let $K = \mathbb{Q}_2$, let $L = \mathbb{Q}_2(\sqrt[3]{2}, \zeta\sqrt[3]{2}, \zeta^2\sqrt[3]{2})$ be the splitting field of $X^3 - 2$, where $\{1, \zeta, \zeta^2\} = \mu_3$.

We have $[L : K] \le 3! = 6$. $\mu_3 \subset L$ so $k_L \supset \mathbb{F}_2(\mu_3) = \mathbb{F}_4$, so $f_{L/K} \ge 2$ and $2 \mid f_{L/K}$. $\sqrt[3]{2} \in L$ so $e_{L/K} \ge 3$ and $3 \mid e_{L/K}$. But now $[L : K] = f_{L/K} e_{L/K}$, so these are in fact equalities.

$$\begin{tikzcd}
L \arrow[d, dash, "3"'] \arrow[d, dash, bend right=60, "I=C_3"'] \arrow[dd, dash, bend left=60, "G=S_3"] \\
\mathbb{Q}_2(\mu_3) \arrow[d, dash, "2"'] \arrow[d, dash, bend right=60, "\mathrm{Gal}\,\mathbb{F}_4/\mathbb{F}_2 \cong C_2"'] \\
\mathbb{Q}_2
\end{tikzcd}$$

Note 10. $\sqrt[3]{2}$ is a uniformiser of L. $0, 1, \zeta, \zeta^2$ are representatives for $\mathcal{O}_L/M_L$.

$$\mathcal{O}_L = \{a_0 + a_1\sqrt[3]{2} + a_2\sqrt[3]{2}^2 + a_3 \cdot 2 + \cdots : a_i \in \{0, 1, \zeta, \zeta^2\}\}$$

is a free $\mathbb{Z}_2$-modulo of rank 6 with basis $1, \zeta, \sqrt[3]{2}, \zeta\sqrt[3]{2}, \sqrt[3]{2}^2, \zeta\sqrt[3]{2}^2$, and $\mathrm{Gal}(L/K)$ permutes the elements — this is not true in general.

$G = S_3$ is the set of permutations of $\{\sqrt[3]{2}, \zeta\sqrt[3]{2}, \zeta^2\sqrt[3]{2}\}$, i.e., the maps

$$\begin{aligned} \sqrt[3]{2} &\mapsto \{\sqrt[3]{2}, \zeta\sqrt[3]{2}, \zeta^2\sqrt[3]{2}\} \\ \zeta &\mapsto \{\zeta, \zeta^2\} \end{aligned}$$

$I = C_3$ is the set of even permutation, i.e., the maps

$$\begin{aligned} \sqrt[3]{2} &\mapsto \{\sqrt[3]{2}, \zeta\sqrt[3]{2}, \zeta^2\sqrt[3]{2}\} \\ \zeta &\mapsto \zeta \end{aligned}$$

Note 11. $\mathrm{Gal}(L/K) \to \mathrm{Gal}(k_L/k_K) = \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2) = \{\iota, (\zeta \leftrightarrow \zeta^2)\}$. On

$$\mathcal{O}_L = \{a_0 + a_1\sqrt[3]{2} + a_2\sqrt[3]{2}^2 + a_3 \cdot 2 + \cdots : a_i \in \{0, 1, \zeta, \zeta^2\}\}$$

elements of $G \setminus I \cong S_3 \setminus C_3$ act non-trivially on a_0, and non-identity elements of C_3 act non-trivially on $\mathcal{O}_L/M_L^2$.

More generally, we let

$$G_i = \{\sigma \in G : \sigma x \equiv x \mod M_L^{i+1}\ \forall x \in \mathcal{O}_L\}$$

be the ith *ramification group*. Then

$$G \supset G_0 \supset G_1 \supset \cdots$$

and $G_0 = I_{L/K}$. In the example above,

$$S_3 = G \supset G_0 = C_3 \supset G_1 = \{\iota\} \supset G_2 = \{\iota\} \supset \cdots$$

Note that

$$G_i = \ker\bigl(G \hookrightarrow \mathrm{Aut}\,\mathcal{O}_L \to \mathrm{Aut}\,\mathcal{O}_L/(\pi_L)^{i+1}\bigr)$$

so that G_i is normal in G for all $i \geq 0$.

Theorem 13.1. Let L/K be Galois, $\pi = \pi_L$ be a uniformiser, $v = v_L$ be normalised and $G = \mathrm{Gal}(L/K)$.

(i) For $\sigma \in I$, $\sigma \in G_n$ if and only if $v(\pi - \sigma\pi) > n$.
(ii) $\bigcap_{n \geq 0} G_n = \{\iota\}$.
(iii) If we write, for $\sigma \in G_n$,

$$\begin{aligned} \sigma\pi &= \alpha_\sigma \pi && n = 0 \\ \sigma\pi &= \pi + \alpha_\sigma \pi^{n+1} && n \geq 1 \end{aligned}$$

then $\sigma \mapsto \bar{\alpha}_\sigma = \alpha_\sigma \mod M$ defines an embedding

$$\begin{aligned} G_0/G_1 &\hookrightarrow k_L^* && n = 0 \\ G_n/G_{n+1} &\hookrightarrow (k_L, +) && n \geq 1 \end{aligned}$$

(iv) If $\mathrm{char}(k_K) = p$ then G is the unique p-Sylow subgroup of $I = G_0$. If $\mathrm{char}(k_K) = 0$ then $G_1 = \{\iota\}$.

Proof. Replacing K by K', the maximal unramified extension in L, we may assume that L/K is totally ramified and $I = G$.

(i) The 'if' direction is clear by definition. For the 'only if' direction, recall that $\mathcal{O}_L = \mathcal{O}_K[\pi]$.

(ii) If $\sigma \neq \iota$ then $\sigma\pi \neq \pi$ because $L = K(\pi)$, hence $\sigma \notin G_n$ for sufficiently large n.

(iii) Let $n = 0$ and take $\sigma, \tau \in G_0 = I = G$. Writing

$$\sigma\pi = \alpha_\sigma \pi, \tau\pi = \alpha_\tau \pi$$

we see that

$$(\sigma\tau)\pi = \sigma\tau\pi = \sigma\alpha_\tau\pi = (\sigma\alpha_\tau)\alpha_\sigma\pi \equiv \alpha_\sigma\alpha_\tau\pi \pmod{\pi^2}$$

as $\sigma\alpha_\tau \equiv \alpha_\tau \pmod{\pi}$ for $\sigma \in I$. So $\sigma \mapsto \bar{\alpha}_\sigma$ is a group homomorphism $G_0 \to k_L^*$. Clearly, $\ker(\sigma \mapsto \bar{\alpha}_\sigma) = G_1$ by part (i).
Now let $n \geq 1$. Similarly,

$$\begin{aligned}
(\sigma\tau)\pi &= \sigma(\pi + \alpha_\tau \pi^{n+1}) \\
&= \sigma(\pi(1 + \alpha_\tau\pi^n)) \\
&= (\pi + \alpha_\sigma\pi^{n+1})(1 + \underbrace{\sigma(\alpha_\tau\pi^n)}_{\equiv \alpha_\tau\pi^n \mod \pi^{n+1}}) \\
&\equiv (\pi + \alpha_\sigma\pi^{n+1})(1 + \alpha_\tau\pi^n) \pmod{\pi^{n+2}} \\
&\equiv \pi + (\alpha_\sigma + \alpha_\tau)\pi^{n+1} \pmod{\pi^{n+2}}
\end{aligned}$$

so here $G_n \to (k_L, +)$ via $\sigma \mapsto \bar{\alpha}_\sigma$ and by part (i) the kernel is G_{n+1}.

(iv) If $\mathrm{char}(k_L) = 0$ then $(k_L, +)$ has no finite subgroup, so for all $n > 0$, $G_n/G_{n+1} = \{\iota\}$, and hence $G_1 = \{\iota\}$.
If $\mathrm{char}(k_L) = p$ then

$$G_n/G_{n+1} \hookrightarrow (k_L, +)$$

is a $\mathbb{F}_p$-vector space. Thus G_1 is a p-group.
$G_1 \triangleleft G_0 = G$, G_1 is a p-group. By part (iii),

$$G_0/G_1 \hookrightarrow k_L^*$$

But k_L is a field of characteristic p so has no elements of order p, so G_0/G_1 has order coprime to p. It follows that $G_1 \triangleleft G_0$ is its p-Sylow subgroup, and $G_1 \triangleleft G_0$ is normal hence unique. $\square$

Definition. G_1 is the *wild interia group.* G_0/G_1 is the *tame inertia group.* L/K is *tamely ramified* if $G_1 = \{\iota\}$ and *wildly ramified* otherwise.

We have the following picture for $\mathrm{char}(k_K) = p$.

$$
\begin{array}{ccc}
\{\iota\} & & L \\
| & & | \\
\vdots & & \vdots \\
| & & | \\
G_2 & & K''' \\
| & & | \\
G_1 & & K'' \\
| & & | \\
I = G_0 & & K' \\
| & & | \\
G & & K
\end{array}
$$

where

- K/K' is maximal unramified, $\mathrm{Gal}(K'/K) = G/G_0 = \mathrm{Gal}(k_L/k_K)$ and this is cyclic if K is local.
- Further, K'/K'' is totally tamely ramified, $\mathrm{Gal}(K''/K') \hookrightarrow k_L^*$, which is the tame inertia group, cyclic of order coprime to p.
- Finally, L/K'' is totally widly ramified, $\mathrm{Gal}(L/K'')$ is the wild inertia group; it is a p-group and quite complicated in general.

Corollary 13.2. G_n/G_{n+1} is abelian for all $n \geq 0$.

Corollary 13.3. I is soluble. If L/K is local then $\mathrm{Gal}(L/K)$ is soluble.

Corollary 13.4. If $\mathrm{char}(k_K) = 0$, e.g., $K = \mathbb{Q}((t))$ or $\mathbb{C}((t))$, then every extension of K is tamely ramified.

13.1 Structure of Tamely Ramified Extensions

Lemma 13.5. Let L/K be Galois, totally and tamely ramified of degree n. Then

(i) $\mu_n \subset K$ since $C_n \cong \mathrm{Gal}(L/K) \hookrightarrow k_K^*$;
(ii) there exists a uniformiser π of K such that $L = K(\sqrt[n]{\pi})$, which follows from Kummer's theorem.

Proof. Exercise. □

Example (S_3-Extensions of $\mathbb{Q}((t))$)**.** Let $K = \mathbb{Q}((t))$. We describe all Galois extensions L of K with $\mathrm{Gal}(L/K) \cong S_3$. Let $\pi_K = t$, a uniformiser of K, and $k_K = \mathbb{Q}$. Recall that this is perfect as it has characteristic 0. Let $k_L = F$, some number field.

Case 1. Suppose L/K is unramified. Then $L = F((t))$ for some S_3-extensions $F/\mathbb{Q}$.

Case 2. Suppose L/K is ramified. $I \triangleleft S_3$, $I \neq \{\iota\}$ and I is cyclic (cf. tame inertia). Thus $I = C_3$ and

$$G/I \cong C_2 \cong \mathrm{Gal}(F/\mathbb{Q})$$

and

$$\begin{tikzcd} L \arrow[d, dash, "3"', "\text{totally ramified}"] \\ K' = F((t)) \arrow[d, dash, "2"', "\text{unramified}"] \\ K = \mathbb{Q}((t)) \end{tikzcd}$$

with $[F : \mathbb{Q}] = 2$.

L/K' is Galois, totally and tamely ramified so, by the previous lemma, $\mu_3 \subset \mathbb{F}^*$ and hence $F = \mathbb{Q}(\mu_3) = \mathbb{Q}(\sqrt{-3})$ as this is the only quadratic extension of $\mathbb{Q}$ containing the cube roots of unity.

By the lemma,

$$L = K'(\sqrt[3]{f})$$

for some $f \in \mathbb{Q}(\mu_3)[[t]]$ of the form $f = ct + O(t^2) = ct(1 + \cdots)$. $1 + O(t)$ is a cube in K'. To see this, either apply Hensel's Lemma or write $(1 + \cdots)^{1/3} := \sum_{n=0}^{\infty} \binom{1/3}{n} (\cdots)^n$. So in fact

$$L = \mathbb{Q}(\mu_3)((t))(\sqrt[3]{ct})$$

for some $c \in \mathbb{Q}(\mu_3)$.

Exercise 10. By considering $\sqrt[3]{\tilde{c}t} \in L$, $c = \alpha + \beta\sqrt{-3}$, $\tilde{c} = \alpha - \beta\sqrt{-3}$ prove that in fact $L = K'(\sqrt[3]{ct})$ with $c \in \mathbb{Q}$.

This gives as the final answer that the S_3-extensions of $K = \mathbb{Q}((t))$ are

(i) $F((t))$ with $F/\mathbb{Q}$ an S_3-extension;
(ii) $\mathbb{Q}(\mu_3)((t))(\sqrt[3]{ct})$, $c \in \mathbb{Q}$, i.e., splitting fields for $X^3 - ct$ over K.